Concrete Foundations for Integers & Linear Algebra

Previously offered as Exploring Integers and Algebra

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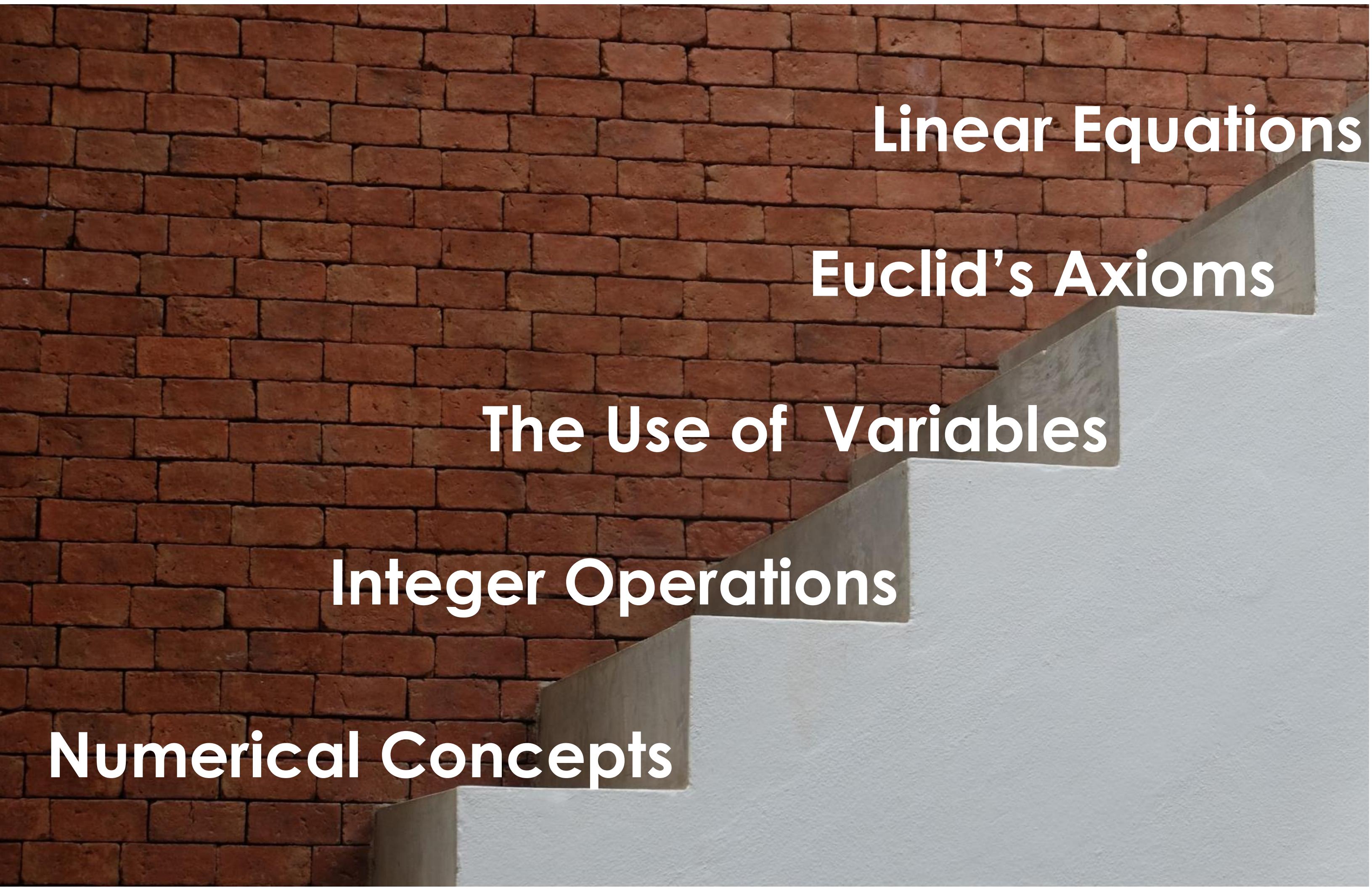

Third Edition

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Concrete Foundations in Integers and Algebra

Communicating Complex Concepts Concretely

A Few Thoughts About Integer Operations and Pre-Algebra

When I ask Montessori teachers and parents what first drew them to Montessori, the response is *almost* universally related to the beautiful math materials that so intentionally reveal the hidden patterns in mathematics. Representing abstract concepts with concrete materials deciphers mathematical mysteries to children and adults alike in a way that mere manipulation of symbols (numbers) according to a set of rules never will. This is more than a romantic idea. It is a principle of mathematics education that is well supported by current neuroscientific research.

A Bit of Neuroscience

PET scans reveal that there is a discrete part of the brain that is responsible for *subitization,* the rapid, accurate judgment of absolute or relative quantity. It is what enables a person to glance at a quantity of 3-4 items and know how many are without counting them. It is also what enables us to look at two different quantities of an object and just know which quantity is greater, again without counting. Subitizing is generally limited to small quantities. Where there are more than about 4 items, subitization fails us and we must count. Montessori materials extended our ability to subitize through the use of color. Consider that we can look at a brown bead bar and know that there are 8 beads without having to count them!

A second discrete area of the brain is responsible for *spatial perception*, another important concept in mathematics. Nowhere is spatial perceptions impact to understanding mathematics more well-developed than in Montessori, where children experience the geometry of numbers as they are developing their first ideas about numeration and quantity.

Neuroscience tells us that these two areas of the brain are located very close to the area of the brain responsible for refined hand control. Recall that Montessori said, "The human hand allows the mind to reveal itself." Neuroscience supports that assertion.

But neither subitizing nor spatial perception is true mathematics. To treat larger quantities, we need to associate the quantity that we subitize or estimate or count with a symbol or concept. This requires language, which is on the opposite side of the brain. Benedetto Scoppola, an Italian researcher in the field of mathematics education, summarizes the research this way: **correct perception of mathematics happens when these two very distant areas of the brain are forced to communicate and to work together.** This is precisely what happens every time we use materials with parallel recording.

PET scans also show that math anxiety happens when all of mathematical knowledge resides in the linguistic/symbolic portion of the brain. The connection with the area that perceives approximate quantity and spatial relationships has been lost, probably due to excessive emphasis on memorizing algorithms and formulas rather than experiencing and understanding concepts. Mathematical memories that are built in conjunction with materials are more durable than those for which we memorize algorithms.

Approaching Algebra Concretely
In many cases, concrete representations of integers and algebraic processes are not included or are treated superficially in Montessori teacher training. This can mistakenly lead even trained Montessorians to the belief that teaching these concepts is traditionally and abstractly is a best practice. Fortunately, as the following lessons illustrate, that is not the case.

There are two themes that run though all of these lessons
1) Present *all* new learning concretely first (but you knew that)
2) Never *tell* a child “the rule” – lead him on a path of self-discovery (auto-education).

Frequently, in these write-ups, you will be cautioned against verbalizing a pattern – the pattern that most of us were taught in school. Instead of an adult saying, “subtracting a positive number is the same as adding a negative number”, carefully structured activities will guide the child to discover that truth for himself. Rather than an adult saying, “Isolate the variable”, concrete experiences will allow him to reason out the proper sequence of steps that will enable him to accurately evaluate an expression.

Some Personal Experience
Is this insistence on concrete presentations “splitting hairs” at this level of mathematics? How important is self-discovery and auto-education with higher math?

It has been my experience that there are fundamental concepts in math that, if fully mastered, open the doors to greater and greater mathematical adventures. If these truths are taught by rote practices rather than by being deeply understood, there are lasting ramifications. The truths become slippery – here today, gone tomorrow. Rather than bolstering the child’s confidence for the ascent to greater adventures, they sow seeds of doubt that will later grow into full-fledged barriers. Children encounter two of these “gateway concepts” early on: the hierarchy of numbers and math facts. Those who are weak on one or both of these areas often struggle with operations work. For algebra and beyond, that “gateway concept” seems to be integer operations. If a budding mathematician can confidently perform all four operations with signed integers and can understand the order of operations as applied to algebraic expressions, all else remains possible. We ensure this confidence when we root everything in experience – large scale and small scale – before taking it to paper and pencil. Buoyed by this success, children can explore everything that linear algebra has to offer with confidence, and can happily look forward to the next levels of challenge: trigonometry, advanced algebra, and even calculus!

Instituting these Lessons in the Classroom
The lessons and activities included here have been intentionally designed to lead children from a concrete experience to abstract understanding for each concept – no intuitive leaps needed! Children with prior experience with negative integers and those who seem to approach math almost instinctually may not need the level of granularity in this sequence. For those children, you will find suggestions as to which lessons might be combined or skipped, distinguishing them from lessons that are needed to lay the groundwork for concepts that are to come. Several lessons also have extensions for those children who are always wanting more challenge.

Most classrooms and schools that I visit choose to introduce integer concepts after children have had significant success with fractions, decimals, percents, and various other mathematical concepts. While this *does* provide a rich background for the study of integers, success in these areas is not a bona fide prerequisite for success with integer concepts and operations. In point of fact, children who have little more than strong number sense and a solid understanding of whole number addition and subtraction can be quite successful with integer addition and subtraction when presented concretely. For the elementary teacher, the significance of this is the impact it might have on the child who has been working diligently to master fractions or decimals with little success. This same child might find integer operations refreshingly straightforward. It might just be the *taste of success* in math that will spur this child on to greater heights!

A Bit of Housekeeping

Throughout these lessons, when writing signed integers, I have chosen to use superscript signs rather than parentheses. Compare these three expressions:

$(+5) + (-4) = (+1)$ $\qquad$ $+5 + (-4) = +1$ $\qquad$ ${}^{+}5 + {}^{-}4 = {}^{+}1$

All three are accurate representations of the same relationship. My preference is to avoid the use of parentheses to bracket signed integers whenever possible. This stems from a concern that when introducing Order of Operations, where parentheses always mean "do me first," the dual meaning of the parentheses will muddy the waters for a time. However, the other two representations are not incorrect and may serve you well in the classroom. Ultimately, the choice of how to use parentheses is left to the discretion of the guide.

Negative Numbers and Other Numerical Concepts

Relative Magnitude of Positive and Negative Numbers

Materials:

Diagram of thermometer showing temperatures above and below 0º (ºF or ºC)
Tickets for various temperatures (in this example, 45ºF, 35ºF, 15ºF, 0ºF, -5ºF)
Red and black tokens or arrows
White board and red and black dry erase markers

Direct Aim: to use children's life experiences to firmly ground them in the meaning of signed numbers and their relative magnitude
To introduce children to the extended number line (positive and negative numbers)

Indirect Aim: to prepare children to work with positive and negative integers
Neuroscience Note: Research shows that humans conceptualize quantity and numeration quite linearly – like a number line. For that reason, the first concrete exposure to signed integers in this album is through a thermometer. A thermometer turned on its side is a number line with positive and negative integers.
Compare that to other examples of signed integers that are often found in textbooks, like earning and spending money, or gaining or losing yards in an American football game. These analogies, while an accurate representation of <u>adding</u> signed integers, do not lend themselves association with a number line – they do not help children understand the meaning <u>and relative magnitude</u> of signed numbers.

Once concept of the meaning of negative numbers and the relative magnitude of signed numbers is internalized, other analogies like football and money can be appropriately used to illustrate early operations if supplemental practice is desired.

Prerequisites: significant success with whole numbers and whole number operations

Presentation:

1. Ask a volunteer to determine the temperature outdoors at the beginning of the lesson. For the purpose of this write-up, we will assume that the outdoor temperature is 45ºF.

2. Show the demonstration thermometer. Affix the 45ºF to the left side of the thermometer. Ask the children what would happen to the temperature if it got windy and cloudy. <It would get colder.>

3. Show the ticket reading 35ºF. Invite a child to affix the ticket on the left side of the demonstration thermometer. Invite discussion about what it would be like outside at 35ºF. If there was precipitation, would it more likely be rain or snow?

4. Lay out the tickets for 15ºF and 0ºF. Invite a volunteer to create a brief story using those two temperatures while another volunteer places those two tickets on the left side of the demonstration thermometer.

5. Ask what would happen if it got even colder. "There is no number lower than 0, less than 0, is there?" <Most children are aware of sub-zero temperatures even if they have not experienced them personally.>

6. Show the -5ºF ticket. If the temperature were to drop by 5º, it would be represented as -5º. Use the terms "5 below zero" and "negative 5" interchangeably. Explain that in mathematics, we make a lot of use of negative numbers, like this temperature that is 5 degrees below zero.

7. Show the 1/2 ºF ticket. Ask a volunteer to show where that temperature would register on the thermometer if we were measuring that carefully. Discuss that all of the proper fractions that they have been studying for years fit between 0 and 1.

8. Show the -1/2 ºF ticket. Ask a volunteer to show where that temperature would register on the thermometer if we were measuring that carefully. Discuss that there are negative fractions, too! All negative proper fractions fall between -1 and 0.
Note: this is a common point of confusion among people who are taught abstractly. Positive proper fractions (and their decimal equivalents) are numbers between 0 and 1. Negative numbers are numbers less than zero. This is not a concept that will be utilized in the following lessons – we will be working exclusively with signed <u>integers</u>. This concept is just for refining children's internal number line to avoid future confusion.

9. Temperature is a measurement of heat. Ask which temperature ticket shows the most amount of heat? <45ºF> Which temperature indicates the least amount of heat? <-5ºF>

10. We have been talking about positive and negative temperatures. Often in math, we work purely with numbers without units. This is also true with positive and negative numbers. When we are working with pure numbers, we show these numbers on a number line.

11. Turn the thermometer with tickets 90º so that the sub-zero temperatures are to the children's left and the warmer temperatures are to their right. Cut a strip of cash register tape roughly equal in length to the thermometer. Lay out the cash register tape parallel to the thermometer, closer to the children than the thermometer. Weight the ends if needed.

12. Create a number line with the same scale as the thermometer, annotating with pure numbers (no units) every 10. Draw the positive side in black and the negative side in red. Bring out the container with red and black tokens or arrows.

13. Ask children to place a black token at the number 50 and another at the number 25. Ask which number is the larger. <25>

14. Remove the tokens and ask children to place a black token at the number 20 and a red token at the number -10. Ask which number is the larger <20>

15. Remove the tokens and ask children to place a red token at the number -5 and another token at the number -15. Ask which number is the larger <-5>
Note: This is the first point at which the children are likely to balk. If they do so, return the thermometer to the original orientation and revisit the concrete concept: temperature is a measure of heat. Which temperature has more heat?

16. Remove the tokens and ask children to place a red token at the number -15 and both a red and a black token at the number 0, signifying that 0 is neither positive nor negative. (Alternatively, use a white or clear token.) Ask which number is the larger < 0 >.

17. Repeat this process until the children show confidence in their answers.

18. (Re)introduce the symbols > and <. If needed, tell the Greedy Goose story – the greater than and less than symbols make up the beak of the Greedy Goose. He always wants to gobble the greater number – the number with a higher value.

19. Use the white board to write a variety of number sentences:

    -9 ______ 4

    Ask the children to complete the number sentence with a < or > (in black).
    Note: Write positive numbers in black and negative numbers in red. (It is not necessary to use the + symbol to annotate positive numbers; the color difference should be a signal AND the convention is that unsigned integers are positive.)

Follow-up and Control of Error: Children's work should reinforce the relative magnitude of signed integers. A half-sheet of problems and its Control of Error follows. Note that this includes problems with equivalent fractions and with conversions between fractions and decimals, just to keep those concepts in play. A blank table is also provided so that teachers can customize the work to the experience level of the students, either to eliminate fractions and decimals or to include the concept of absolute value (from the Extension).

NOTE: The concept of relative magnitude is one that may need repetition to internalize. Race to Ten (which follows) is a game that encourages further practice with the concept. Since it is a game to be played with a friend, repeating it has a greater appeal than giving yet another paper-pencil activity. It is suggested that children who have completed the paper-pencil follow-up activity can be encouraged to play the game. It is left to the discretion of the teacher whether to provide instructions for the game at the conclusion of the lesson or to wait until a few children have completed the follow-up

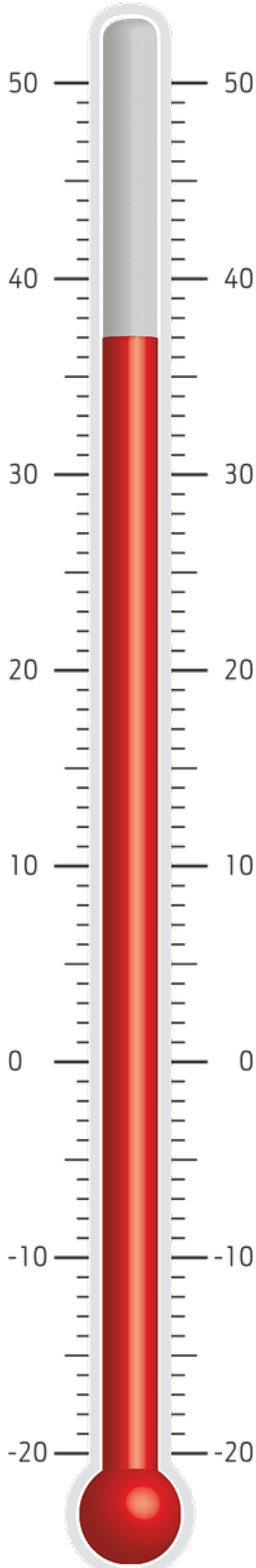
Extension: Absolute Value
The absolute value of a number is the magnitude of the number without regard to its sign. The absolute value of a number is its distance from zero irrespective of distance.

It is ALWAYS a positive number. (Consider that in real life, measurement of distance is always positive – there are no negative tape measures!)

It is annotated with vertical bars on either side of the numeral and sign.
$|5|$ = "the absolute value of 5" = the magnitude of 5 = 5
$|-5|$ = "the absolute value of -5" = the magnitude of -5 = 5

Or in algebraic terms:
$|-x| = x$ and
$|x| = x$
so $|-x| = |x|$ (A number and its opposite are the same distance from 0.)

This can be a great way to "up the ante" with magnitude of numbers work. For some children, the concept of relative magnitude will make more sense when they learn about the absolute value (absolute magnitude) of a number. For others, it will muddy the waters. If this concept is deferred for now, there will be another opportunity to teach this concept a bit later in the sequence, when children work with the Snake Game.



45°F

35°F

15°F

0°F

-5°F

½ °F

-½ °F

Relative Magnitude of Positive and Negative Numbers

Complete each number sentence with <, >, or =. The first one is done for you.

-25	<	**5**
50		**-50**
-12		**12**
0		**-87**
4/5		**-4/5**
-1		**-1/3**
.085		**.090**

-25		**-55**
-50		**0**
1/2		**-1/2**
0		**-.87**
3/4		**.75**
-1/2		**-0.5**
.25		**-1/2**

Relative Magnitude of Positive and Negative Numbers

Complete each number sentence with <, >, or =. The first one is done for you.

-25	<	**5**

Control of Error for **Relative Magnitude of Positive and Negative Numbers**

Complete each number sentence with <, >, or =. The first one is done for you.

-25	<	**5**
50	>	**-50**
-12	<	**12**
0	>	**-87**
4/5	>	**-4/5**
-1	<	**-1/3**
.085	<	**.090**

-25	>	**-55**
-50	<	**0**
1/2	>	**-1/2**
0	>	**-.87**
3/4	=	**.75**
-1/2	=	**-0.5**
.25	>	**-1/2**

Control of Error for **Relative Magnitude of Positive and Negative Numbers**

Complete each number sentence with <, >, or =. The first one is done for you.

-25	<	**5**

Euclid's Laws: Solving Linear Equations

NOTE: Linear equations have a single variable

Euclid's 2nd and 3rd Axioms

Materials:

Pan balance and weights (ideally, pans can be detached from the balance)
Positive and negative bead bars from Negative Snake Game

- fused nylon type with no wire are preferable, but any matched set will work
- do not combine types of bead bars (weight discrepancies will result)

An opaque small box with a lid

Direct Aims: to introduce methodically solving linear equations using Euclid's laws

Indirect Aim:
Preparation for more advanced algebra work
Establishing the procedural memory of methodically solving problems showing steps
Reinforcing principles of integer operations
Reinforcing inverse operations: addition and subtraction & multiplication and division

Prerequisites:
Extensive experience with integer operations
Experience with math facts as special cases (i.e. 6 + ☐ = 9) is helpful, but not required.
Experience with special cases expressed algebraically (6 + x = 9) is helpful, but not required.

Presentation

1. Bring out the pan balance. Demonstrate that when equal weight is in each of the two pans, the pans are equidistant from the tabletop and the indicator, if there is one, is vertically aligned. The two pans are in balance with one another.

2. Demonstrate using the pan balance to illustrate addition problems. Write 6 + 3 = x
   Place a 3-bar and a 6-bar in the leftmost pan (child's perspective). Ask what bead bar should be placed in the other pan to balance the two pans. <a 9-bar>.
   Show with the balance that placing a 6-bar and a 3-bar in the leftmost pan balances out with a 9-bar in the rightmost pan
   Complete the problem: 6 + 3 = x; **x = 9**.

3. Draw an analogy between the parts of the problem and the scale. The = is the balance point, the fulcrum on the balance. Consider the = sign to be the center of the equation; the expressions on either side, whether an expression (6+3) or a single numeral (9) are the contents of the two pans.

4. Illustrate an equation where the variable is one of the addends. Write 9 + x = 18 on the board.
   Place a 9-bar in one pan and a 10-bar and an 8-bar in the other pan. Ask what needs to happen to balance the pans. <A 9-bar should be added to the first pan.> Complete the action and solve for x.

   9 + x = 18; **x = 9**

5. Once the children find this intuitive and straightforward, demonstrate a problem with negative numbers such as $^{-}8 + 4 = a$. The children will know from prior experience that $a = {}^{-}4$, but demonstrating this is tricky. Since both the yellow $^{+}4$-bar and the two-tone grey $^{-}8$-bar have weight, placing them both in the same pan will be additive. Together, they will weigh the same as 12 beads. We know that the correct answer is $^{-}4$, but placing a $^{-}4$ bead-bar in the pan will not cause the scale to balance.

6. Explain that we know if bead bars **represent** positive or negative numbers because of their color. Changing the color of the bead bars does not change their weight. Because of this, we can't use a real balance.

   For many lessons now, we have been using our eyes and our brains to interpret some pictures or colors to mean weight and other pictures or colors to mean lift. We need a balance that can use our interpretations of pictures or colors – a **representative** balance or a **virtual balance** - to do this type of problem.

7. Remove the two pans from the balance and set them on the rug. Explain that we are going to use our brains to decide if the two pans are in balance with one another. They will make a virtual balance.

   Place the $^{-}8$-bar and $^{+}4$-bar in the pan to children's left and an opaque box in the pan to the children's right. Explain that the opaque box represents our unknown, *a*. Place a $^{-}4$ bar in the opaque box. Again, ask if the two pans are now in balance. <They are: $a = {}^{-}4$>

8. Demonstrate a somewhat trickier problem where the variable is one of the addends rather than the sum, such as $9 + a = {}^{-}7$. Place a 9-bar into the virtual pan on the left and a $^{-}7$ bar into the virtual pan on the right. Place an opaque box onto the pan on the left, saying, "this box represents *a.* If we have the right amount in the box *a,* the two pans are in balance. <The children will likely know that the missing addend is $^{-}16$. If they do, congratulate them on their prowess with integers. Tell them that more complicated problems are coming where they won't be able to do the problem in their heads as easily.>

9. Ask them to help you demonstrate a method of solving the problem for *a* when they can't just figure it out in their heads. To learn what goes in the box, we want to end up with the variable (represented by the box) to end up all by itself in one pan with bead bars in the other pan. We can then say *a* (the pan on the left) = the number represented by the bead bars (the pan on the right). *NOTE: be sure that the children understand this goal before proceeding.*

10. "The 2 pans are in perfect balance. We want the 9-bar in the left pan to go away so we will be left with only the box. What would happen to our balance if we just took the 9-bar away?" <The pans would be out of balance! We can't just throw quantities away!>

11. Take all suggestions of how to make the 9-bar "go away". If there are no suggestions on how to make it go away, ask if there are ideas about how to neutralize the 9-bar. (If there are still no suggestions, remind that we are talking about 9 weights. How can we neutralize 9 weights? <With 9 balloons!>)

12. If we add $^{-}9$ to 9 on the left pan, those two bead bars neutralize each other, resulting in 0. If we add $^{-}9$ to the pan on the left, what must we do to the pan on the right to keep the pans in balance? <We must add $^{-}9$ to that pan as well!>

13. Add a $^{-}9$-bar to each pan simultaneously. (Still in "balance"!) Then simplify:
    Take the $^{+}9$-bar and the $^{-}9$-bar out of the pan on the left, saying "9 - 9 = 0".
    Sum the bead bars in the pan on the right, saying "$^{-}9 + {}^{-}7 = {}^{-}16$."

14. Note that what remains shows that $a = {}^{-}16$. Verify that this is correct by plugging in the answer: $9 + {}^{-}16 = {}^{-}7$.

15. Let's record what we did:

    $9 + a = {}^{-}7$ *this is our original equation*
    $\underline{{}^{-}9 \quad = {}^{-}9}$ *we added negative 9 to both sides*
    $(9 + {}^{-}9) + a = {}^{-}16$ *this is what resulted*
    $a = 16$ *when we removed +9 and -9, this was the result*

16. Tell the children that there was a famous Greek who was a well-respected teacher of mathematics in Alexandria, Egypt around 300 BC. He wrote a book called The Elements, which was not only the basis for teaching math in Egypt in his time, but was also the main mathematical textbook all around the globe until the late 19th or early 20th century! Among many other things, Euclid wrote some rules for working with equations, which he called *axioms*. What we just discovered is Euclid's second axiom!

Euclid's Second Axiom: ***If equals be added to equals, the wholes are equals.***

17. Refer to the problem just demonstrated to illustrate Euclid's Second Axiom. We began with two pans that were equal $^{-}9+a$ and $^{-}7$. We know that they were equals because there is an = between the two sides.
    We added $^{-}9$ to both sides of the equality: $^{-}9 = {}^{-}9$, so we added equals to equals.
    This resulted in a new equality: $a = {}^{+}16$

18. Demonstrate and record another problem: $^{-}15 + b = 3$

Action	Record
Place a -10-bead bar, a -5-bead bar and an opaque box in the left pan, and a 3-bead bar in the right pan	$^{-}15 + b = 3$
Add a 10-bar and a 5-bar to each pan	$^{-}15 + b = 3$ $+15 \quad = +15$
Show that, in the left pan, the bead bars sum to 0, leaving only the opaque box, Replace the 3-bar and the 5-bar in the right pan with a brown 8-bar.	$^{-}15 + b = 3$ $\underline{+15 \quad = +15}$ $0 + b = 18$
Remove the bead bars from the left pan	$^{-}15 + b = 3$ $\underline{+15 \quad = +15}$ $0 + b = 18$ $b = 18$

19. Demonstrate and record a problem where the first addend is the variable and it is necessary to *subtract* (add negative bead bars) to isolate the variable: c + 12 = $^{-}5$.

Action	Record
Place an opaque box, a 10-bead bar, and a 2-bead bar in the left pan, and a -5-bar in the right pan	$c + 12 = -5$
Add a -10-bar and a -2-bar to each pan	$c + 12 = {}^{-}5$ $-12 = -12$
Show that, in the left pan, the bead bars sum to 0. Replace the -5-bar and the -2-bar in the right pan with a -7 bar.	$c + 12 = {}^{-}5$ $\underline{-12 = -12}$ $c + 0 = {}^{-}17$
Remove the bead bars from the left pan.	$c + 12 = {}^{-}5$ $\underline{-12 = -12}$ $c + 0 = {}^{-}17$ $c = {}^{-}17$

Remark, “we are using Euclid’s third axiom!

Euclid’s Third Axiom: *If equals be subtracted from equals, the wholes are equals.*

20. Finally, demonstrate a subtraction problem: $d – 25 = 97$ *NOTE: this problem uses the best practice of changing subtraction into adding the opposite.*

Action	Record
Write the original equation on the board. Transform the subtraction problem to an integer addition problem.	$d – 25 = 67$ $d + {}^{-}25 = 67$
Place an opaque box, two $^{-}10$-bead bars, and a $^{-}5$-bead bar in the left pan, and six 10-bars and a 7-bar in the right pan	$d – 25 = 67$ $d + {}^{-}25 = 67$
Add a two 10-bead bars, and a 5-bead bar to each pan	$d – 25 = 67$ $d + {}^{-}25 = 67$ $+25 = +25$
Show that, in the left pan, the bead bars sum to 0. Replace the 7-bar and the 5-bar in the right pan with a 10-bar and a 2-bar.	$d – 25 = 67$ $d + {}^{-}25 = 67$ $\underline{+25 = +25}$ $d + 0 = 92$
Remove the bead bars from the left pan.	$d – 25 = 97$ $d + {}^{-}25 = 97$ $\underline{+25 = +25}$ $d + 0 = 92$ $d = 92$

Follow-up: Children should complete a selection of similar problems. If they feel that they can complete the problems using only paper and pencil, <u>so long as they record the steps</u> (be insistent on this!) and so long as they do so correctly, they can be permitted to do so. Showing the steps is important for two reasons:

- We must be sure that children are using algebraic concepts and Euclid's Axioms and not just getting the "right answer" by some other means that does not reinforce the isolated difficulty of this lesson.
- Children will be held accountable (on standardized testing and in high school algebra classes) for showing their work. It we establish this as a habit now, the practice will come more easily and with less resistance when the needed.

The next lesson has the added complication of a variable with a coefficient. Children will need to be accomplished with applying the first two axioms to be successful with the next lesson.

NOTE: any problems answered incorrectly should be redone with materials and parallel recording.

Control of Error: Checker Card or Teacher.

Super Geeky Teacher note: Euclid's 2nd and 3rd axioms are actually a bit more complicated than illustrated here.
His first axiom, which we omitted, says, "Things which are equal to the same thing are also equal to one another."
In algebraic terms, that is saying that if a = b and c = b, then a = c.
A numerical example is 1+5= 6, and 2+4=6, so 1+5=2+4.

We illustrated the second axiom by adding a constant to both sides. Euclid was not as constrained, saying that as long as what you add to both sides is equal, the whole are equal. In other words, on the first example where we added ⁻9 to both sides, we could have added -9 to one side and ⁻4 and ⁻5 to the other side. Since ⁻9 = ⁻4 + ⁻5, we would have added equals.

His axiom actually states, "if x=y and a=b, then x+a = y+b. This is an unnecessary complication at this stage of the child's learning. It is included here just for the pure geeky joy of it, and to provide a bridge to whatever algebra texts the teacher might choose to access.

Applying Euclid's Second and Third Axioms to Integer Operations

Use solve these algebraic expressions, writing each step as shown in the example.
Remember to appropriately apply Order of Operations!

Example: $-25 + z = 150$

$\underline{+25 \quad = +25}$

$z = 175$

⇒ $^{-}52 + y = -26$ $\quad$ $y =$

⇒ $x - 13 = 100$ $\quad$ $x =$

⇒ $(53 – 10) + w = 0$ $\quad$ $w =$

⇒ $v + 3 \times 5 = 15$ $\quad$ $v =$

⇒ $^{-}173 + u = {}^{-}22$ $\quad$ $u =$

⇒ $173 + t = {}^{-}22$ $\quad$ $t =$

⇒ $s + (2 + 3)^2 = 2 \times 5 + 5$ $\quad$ $s =$

⇒ $9^2 + 3 \times 2 + r + {}^{-}20 = 87$ $\quad$ $r =$

⇒ $q + 5 \times 4 – 2 = 6 \times 4 - 4$ $\quad$ $q =$

⇒ $^{-}20 + p = 3^2 + 3 \times 4 - 5$ $\quad$ $p =$

Control of Error for Applying Euclid's Second and Third Axioms to Integer Operations

Example: $-25 + z = 150$
$+25 \quad = +25$
$z = 175$

⇒ $^{-}52 + y = -26$
$+52 \quad = +52$
$y = 26$ — $y = \mathbf{26}$

⇒ $x - 13 = 100$
$x + {}^{-}13 = 100$
$+13 = +13$
$x = 113$ — $x = \mathbf{113}$

⇒ $(53 - 10) + w = 0$
$43 + w = 0$
$-43 \quad = -43$
$w = -43$ — $w = \mathbf{-43}$

⇒ $v + (3 \times 5) = 15$
$v + 15 = 15$
$-15 = -15$
$v = 0$ — $v = \mathbf{0}$

⇒ $^{-}173 + u = {}^{-}22$
$173 \quad = 173$
$u = 151$ — $u = \mathbf{151}$

⇒ $173 + t = {}^{-}22$
$-173 \quad = -173$
$t = -195$ — $t = \mathbf{-195}$

⇒ $s + (2 + 3)^2 = (2 \times 5) + 5$
$s + 25 = 15$
$-25 = -25$
$s = -10$ — $s = \mathbf{-10}$

⇒ $9^2 + (3 \times 2) + r + {}^{-}20 = 87$
$(81 + 6 + {}^{-}20) + r = 87$
$67 + r = 87$
$-67 \quad = -67$
$r = 20$ — $r = \mathbf{20}$

⇒ $q + (5 \times 4) - 2 = (6 \times 4) - 4$
$q + 20 - 2 = 24 - 4$
$q + 18 = 20$
$-18 = -18$
$q = 2$ — $q = \mathbf{2}$

⇒ $^{-}20 + p = 3^2 + (3 \times 4) - 5$
$^{-}20 + p = 9 + 12 - 5$
$^{-}20 + p = 16$
$+20 \quad = +20$
$p = 36$ — $p = \mathbf{36}$