Concrete Foundations for Algebraic Operations

Multiplying Binomials

$(2x-3)\,(-5x+4) = -10x^2 + 23x - 12$

and

Factoring Quadratics

$3x^2 - 14x + 8 = (3x-2)(x-4)$

by Betsy A Lockhart

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For information address:
lockhartlearning@gmail.com
30472 Middleton Road, Evergreen, Colorado 80439

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Multiplying Binomials and Factoring Quadratics

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A Few Thoughts Between Friends
or
Algebra, with a Side of Cognitive Neuroscience

The field of cognitive neuroscience has been exploding for the past decade. It has much to say about how the human brain perceives quantities and their symbolic representations, as well as operations work. These findings reinforce many aspects of what Dr. Montessori said over 100 years ago.

Correct Perception of Mathematics
Neuroscience tells us that within the human brain, there specific areas that are responsible for different aspects of mathematics. On one side of the brain, we have areas dedicated to

- spatial perception – the ability to perceive spatial distances including the length, width, and height of objects, distances between objects, and shape, applying it in everyday life (Will my car fit in this parking space?)
- subitization – the instantaneous, unconscious perception of quantity without counting. *Relative subitization* is the ability to compare two quantities and know which is the greater without counting. Many animals are known to have this ability. (Which of these two trees has the most fruit and is, therefore, more worth the effort to climb?). Research is showing that all mammals have this ability, as do some birds, fish, and insects. *Absolute subitization* is the ability to look at a small collection of objects and "just know" how many there are. In humans, this is limited to 3-5 objects.
  Our ability to "just know" haw many there are in a quantity can be extended by pattern. For example, dice and dominoes use a pattern for 5 that is 4 dots at the vertices of an imaginary box with the 5th dot in the center. When we see that pattern, we know it represents 5 without having to count the dots.
  Montessori math extends our ability to look at a quantity of beads and "just know" how many there are through color; we "just know" that there are 8 beads in a brown bead bar.
- refined hand control – the ability to manually manipulate objects to gain information about them sensorially and to arrange objects to create meaning. Dr. Montessori said it this way in The Absorbent Mind, "The hands are the instruments of man's intelligence."

Interestingly, these three areas of the brain are located very close to one another – they make connections naturally and easily. Dr. Montessori did not have functional MRIs and PET scans to see how the brain processes information. But she knew that processing information with real, 3-dimensional manipulative objects was key to learning. In *The Absorbent Mind*, she said, "He does it with his hands, by experience, first in play and then through work. The hands are the instruments of instruments of man's intelligence."

But this is not the whole story. True mathematics is more than just the perception and physical/mental manipulation of quantities. It requires associating quantity with a symbol or concept. This requires language, which is centered on the opposite side of the brain. Benedetto Scoppola, an Italian researcher in the field of mathematics education, summarizes the research this way: *correct perception of mathematics happens when these two very distant areas of the brain are forced to communicate and to work together*. Many Montessori early math materials are dedicated to associating quantity and symbol: red and blue rods with numerals, cards and counters, tens and teens boards, and more. But the importance of associating physical meaning with abstract representations continues well beyond counting and numeration.

Correct perception of mathematics has life-long implications. PET scans show that math anxiety happens when all of a person's mathematical knowledge resides in the linguistic/symbolic portion of their brain. When that happens, the connection to the area that understands the concrete aspects of math has been lost, probably due to excessive emphasis on memorizing algorithms and formulas rather than experiencing and understanding concepts. In short, mathematical memories that are built in conjunction with materials are more durable than those for which we memorize algorithms. In Montessori, every time we solve a problem using materials with parallel recording, we are linking those two distant areas of the brain to produce correct perception of mathematics.

Another interesting bit of research reveals that the human brain perceives quantity much like a number line. Perceiving quantities that are farther from 0 takes a miniscule but measurably longer time than perceiving quantities that are closer to 0. This has implications for introducing signed integers – we need to favor materials and analogies where there is a definitive 0 and a linear progression away from 0. (Football analogies, for example, should be reserved for a bit later in students' studies.)

Approaching Algebra Concretely

The importance of linking concrete and abstract understanding of mathematics extends into algebra. Even people who have a good relationship with arithmetic can lose their footing when representation of quantity goes to a more abstract level. There is often a feeling of loss associated with algebraic expressions that exist just for their own sake; what meaning is there to 5x+3 if we don't know what x is? Why do we simplify equations if we aren't going to solve them? Why are x^2 and 2x not like terms that cannot be combined? After all, they each have a 2 and an x…

Concrete representations of integers and algebraic processes are not included or are treated superficially in most Montessori teacher training. This can lead even trained Montessorians to the mistaken belief that teaching these concepts abstractly is a best practice. It is not. Dr. Montessori's advice was the quite the opposite, saying that instruction should progress from the sensorial/concrete experience stage through the mathematical reasoning/measurement stage to the formulaic stage. (See her writings on math and geometry in Psychoarithmetic and Psychogeometry). Recent findings in the cognitive neuroscience back her up.

Both albums in the Lockhart Learning Concrete Foundations in Algebra series follow Montessori's axiom that instruction progress from the concrete stage through the reasoning stage to the formulaic stage.

- Integers and Early Algebra (the first book in the Algebra series) introduces students to signed integers concretely and linearly, exploring relative magnitude and then ventures into the 4 integer operations (math facts with signed integers). It also shows how to solve basic linear equations with standard Montessori materials, including linear equations with coefficients that are signed integers or fractions.

  $$^{-}4x-3 = 5 \text{ or even } \tfrac{1}{4}\, x - 9 = {}^{-}6$$

  Each section in this album culminates with parallel recording that leads to abstract problem solving.

- Algebraic Operations (this album) extends the students' knowledge, leading them to discover how to multiply algebraic binomials starting with simple problems like (x+4)(x+3) and building through isolated difficulties until students can multiply more complex problems like (5x-2)($^-$2x+3). These lessons also provide a bridge between the concrete and abstract representations of binomial multiplication. Once students are able to multiply binomials to produce a quadratic equation, lessons lead through the reverse process, factoring a quadratic equation into its binomials factor pairs.
  This is lesson sequence of multiplying binomials to produce a quadratic and factoring quadratics back to their composite factors is analogous to the Montessori lesson sequence on squaring and finding square roots; first we learn to build up (squaring/multiplying binomials) and then we learn to factor back (square roots/factoring quadratics).
  Once again, lessons progress through isolated difficulties that build bridges between concrete and abstract factorization.
  Students with prior experience with multiplication and squaring with materials will find these lessons have a very familiar feel to them – they will be making connections to prior learning throughout. Students new to Montessori can easily pick up on the concepts, even though they won't have as many points of contact with prior learning.

Lessons in both albums in the Lockhart Learning Concrete Foundations in Algebra series isolate the difficulties at the most granular level. Students with abundant prior Montessori experience and/or an intuitive and spatial approach to math may progress more rapidly. The guide is always encouraged to gauge students' reception of new concepts during the lesson to decide if there is enough new challenge to keep things interesting. If students are one step ahead of each step of the presentation, consider continuing on to the next lesson.

All of this work is intended feather into the abstract processes that are taught in textbooks and tested in standardized tests, providing a concrete rationale for the abstract rules that they will be expected to follow. Students who complete these lessons will understand the geometry that the algebra is describing. They will be able to use that reasoning to approach algebraic equations with unexpected twists and turns, applying strategies that are based in the physical reality of geometry, not just on the rules that math teachers told them had to be followed to get The Right Answer. And who knows; teaching these lessons may even begin to heal math anxiety for some guides!

If you have any questions regarding these lessons, please feel free to contact me at lockhartlearning@gmail.com .