Squaring
and
Cubing
and
Rooting
(oh my!)

by

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Third Edition

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Prologue: Thoughts Before Beginning

"Mathematical Minds" and the Importance of Discovery

Maria Montessori believed that all humans were blessed with a "mathematical mind". By this she did not mean that all people, properly taught, would excel in computational arithmetic and geometry. Rather, she believed that humans had a natural affinity for order, precision, and exactness. She said that this reasoning mind had a natural ability to orient itself, communicate with others, and use imagination in problem solving. Certainly, all of these traits are assets in the systematic study of mathematics. The beautiful and intuitive Montessori math materials and methods are specifically designed to capitalize on and reinforce these human tendencies in the pursuit of mathematical understanding and fluency. Mathematics in Montessori is about providing students with the means to *discover* relationships and key truths.

The alternative to discovering relationships is memorizing algorithms. Certainly, humans can learn mathematical operations algorithmically; in fact, that is the way most of us learned math. So why not "cut to the chase"? Why do we painstakingly lay bread crumbs for children, leading them through concrete experiences before going to paper and pencil? This could be the topic of a hundred-page dissertation! Here are but two important reasons.

- Relationships (mathematical and others) naturally build upon themselves in a multi-directional / multi-dimensional sense. A concrete approach allows for discovery of these relationships, unlike linear-sequential, product-oriented, algorithm-based knowledge that focuses the mind on producing The Right Answer. Discovery of mathematical truths produces a woven textile of concepts rather than a collection of strands of ideas.
- Learning that progresses from concrete experiences to abstract manipulation of concepts is developmentally appropriate for second plane children who are experiencing a "birth of abstraction". With these children, we can use some abstract processes to augment children's understanding; things like analogies, logic, and compare/contrast activities will expand understanding that is developed concretely and will help children better internalize how one concept relates to another. Please note: this does not mean that second-plane children are ready for instruction that is presented abstractly.

Derailing Discovery

If materials-based, concrete instruction is more effective, why do some well-intentioned Montessorians heavily supplement or replace the method with traditional computation-driven, product focused instruction in upper elementary? Here is what some will say:

- **"The materials don't speak to the kids as they get older."** If these amazing materials are not speaking to the children, what is drowning them out? Often, the attitude of the adults (teachers or parents) communicates the unspoken message that the ultimate goal – the measure of success - is *abstraction* rather than *understanding*. Children want to be successful. If the definition of success is abstraction, children will tend to shed materials more quickly – often too quickly. We must be cautious about our attitudes and words. Phrases like, "When you know your math facts, you won't have to use this material anymore," can creep in and convey an implicit message that may not be serving the children well.
- **External pressure.** Using a counter-cultural teaching method like ours means that what we do is often misunderstood. Outside observers (state / district educators, parents, and others) cannot embrace the totality of our curriculum and method through occasional short-lived

observations; unfortunately, the observers often don't realize that. They believe that they understand Montessori, just as each of the blind men believed that he knew what an elephant was by feeling a portion of the animal. Partially informed (and well intentioned) adults sometimes pressure Montessorians to supplement or replace our methods with methods that they have more experience with – methods that they trust. This becomes more pronounced as the stakes get higher.

- **Lack of overt alignment with standards.** Montessori lessons and materials <u>unequivocally</u> meet and exceed standards. Of this there can be no doubt. Numerous people who have documented alignment with different sets of standards over time. A difficulty arises from the fact that a logical sequence for product and algorithm based instruction does not always match that for instruction based on concrete experiences. Take, for example, fraction operations. Computationally, fraction multiplication is very simple and straightforward: multiply the numerators; multiply the denominators; reduce the product if needed. Computationally, this is far simpler than non-common denominator addition or subtraction, which has many steps involving lots of arithmetic. But conceptually, concretely, multiplication is far more complicated than addition. This is where blending traditional and Montessori approaches gets tricky. If we rigidly adhere to a textbook or standards for scope and sequence, we must know that there will be some times when a concept is next in sequence requires us to take a sidetrack, to lay the foundation concretely.
- **Test results anxiety.** Standardized testing has been a hot topic for a very long time. Here are three fundamental truths about testing:
  - These tests provide valuable information about what a child can do abstractly and independently. If the results of testing provide meaningful, detailed metrics to teachers, it gives us a tool to improve our instruction.
  - Tests, like the humans that design them, are flawed. Sometimes the repercussions of these flaws are small and sometimes they are great. While they can provide valuable information, we must understand what that information is and is not telling us.
  - For the most part, we have little or no control over the choice of tests. These decisions are made well above the classroom level. This means that our time is best spent understanding what is expected of our children and ensuring that they meet those expectations, even though the path they take to mastery is different from their peers in traditional school.

  It is absolutely true that tests sometimes use different language that can completely derail our students if we are unaware and do not provide a translation (for example, *ones* vs. *units*). It is also true that sometimes the way concepts are depicted in testing is just different enough from what our children experience that it can cause confusion about the question being asked. (For example, the standard representation for a right angle is a small square at the vertex of the angle. Compare that to the red arc in the stick box.) Even something as simple as changing from vertical to horizontal orientation can make something that is well known feel unfamiliar.

  All of this means that practicing for standardized tests using test-prep activities that align with the test being given is an important Practical Life activity. Beyond that, it will generate a new way of looking at some mathematical procedures, which actually makes children's math-textile even richer.
- **"Using materials to supplement abstract instruction is the best of both worlds"** There are those who will regard Montessori as a wonderful enrichment opportunity or as a means

of remediation for children who didn't internalize concepts from more traditional methods. Montessori is neither for enrichment nor for remediation. It is a different way of thinking. If we "tack on" Montessori to traditional methods, we have given a child 2 very different ways to look at a single concept, often without allowing the time to fully comprehend either. This is a recipe for disaster.

Preserving Discovery

The lessons in this album are specifically designed to begin each concept in the concrete. Sometimes they use traditional Montessori materials. Sometimes they use other items, like checkers or sidewalk chalk. All are intended to give children a concrete, sensorial experience before computations kick in. Whenever possible, recording parallels concrete experiences to build that all-important bridge linking concrete experiences (applications) to abstract reasoning.

While classic lessons involving Montessori apparatus still constitute the majority in this album, some topics have been added specifically because children are now being held accountable for these concepts at a younger age. Newer lessons on ratio/proportion, percents, statistics/probability, and early Algebra follow Montessori best-practices. They use Montessori and alternative materials as much as possible in the introductory lessons, de-scaffolding to paper-pencil processes as lessons progress. They follow a carefully designed sequence of isolated difficulties. It is hoped that these lessons will provide an alternative to lessons sourced from textbooks or online resources, which by-in-large have a very process-focused, algorithm based approach. It is hoped that this will enable teachers to protect the time and space needed for children to discover concepts rather than consume them.

For many of these new lessons, common core standards that pertain to the lesson have been listed. When a lesson addresses only part of a standard, it is so indicated. **This is not intended to convey that by giving the lessons in this album, all standards will be thoroughly addressed.** It is still the work of the teacher to ensure that students have a reasonable opportunity to meet whatever standards the school follows or sets.

Each of the new lessons comes with a follow-up and a control of error. Sometimes this singular experience will be enough for a child to internalize the difficulties of the lesson. Sometimes further practice will be needed. If further practice is needed, traditional resources <u>may</u> be an appropriate choice to expose children to alternative notations and language. If these resources are used, please ensure that the problem types and the instructions are a good match for the lesson. Here are two things to watch for:

- Ensure that problems provided can be completed using the process taught in the lesson. This is especially important when the lesson is very concrete. Be sure that the problems do not go beyond the limitations of the materials.
- Ensure that example problems in the text or at the top of a worksheet do not reveal something that children have not yet discovered. If the children have not yet discovered the algorithm, please do not expose them to it through follow-up work!

It is hoped that with these new resources, this album will be your primary resource for lessons for years to come, and that you and your children will fall deeply in love with both the elegant simplicity and the intricate interconnectedness of all that is mathematics.

Author's Note:

The Third Edition (2021) of this album adopted the newly (re)ratified pronoun: the "singular *they*". This pronoun use is not new – it's use has been documented as long ago as medieval times (or perhaps older). It was eighteenth-century grammarians who called for *they* and *them* to be reserved for use only in the plural form. It seems that no one knows why this adjustment was made while the same adjustment wasn't made with the pronoun you (also used in both the singular and plural forms - then and now, of course). Perhaps the roiling debate about whether the "casual you" was an acceptable replacement for thou, thee, and thy overshadowed the issue of number.

Whatever the history of these pronouns, the MLA now encourages writers to use the pronouns that the individual being written about has chosen as their personal pronoun. They also acknowledge and encourage the use of *they* as a gender non-specific, generic, third-person singular pronoun…

> "…to refer to a person whose gender is unknown or irrelevant to the context … This use of singular *they*, until very recently discouraged in academic writing and other formal contexts, allows writers to omit gendered pronouns from a sentence like the following:
>
> Each student must find his or her folder before class.
>
> Instead, writers may substitute singular *they*:
>
> Each student must find their folder before class.
>
> Because it lacks grammatical agreement, this use of singular *they* has been considered a less desirable option than revising to use the plural or rephrasing without pronouns. But it has emerged as a tool for making language more inclusive … and the MLA encourages writers to accept its use to avoid making or enabling assumptions about gender.

"How Do I Use Singular They?" *The MLA Style Center*, 22 Mar. 2020, style.mla.org/using-singular-they/.

In addition, in 2019, Merriam-Webster added a definition for *they* as "used to refer to a single person whose gender is intentionally not revealed… used to refer to a single person whose gender identity is non-binary"

"They." *Merriam-Webster*, Merriam-Webster, www.merriam-webster.com/dictionary/they.

Gender-specific pronouns are used sometimes in this album, occasionally in story problems or in reference to particular historical figures.

-Betsy Lockhart (2020)

Advanced Whole Number Operations: Squaring
Stamp Game Materials

Materials:

Two Stamp Game sets
Golden Bead Materials used in "Squaring – Golden Beads"
Two sets of Numeral Cards
Addition, Equals, and parentheses signs
Graph paper
Pencils and colored pencils
Montessori Pattern Cards for the Analysis of Squares

Aims: Introduction to the pattern of squaring binomials
Preparation for extracting the square root

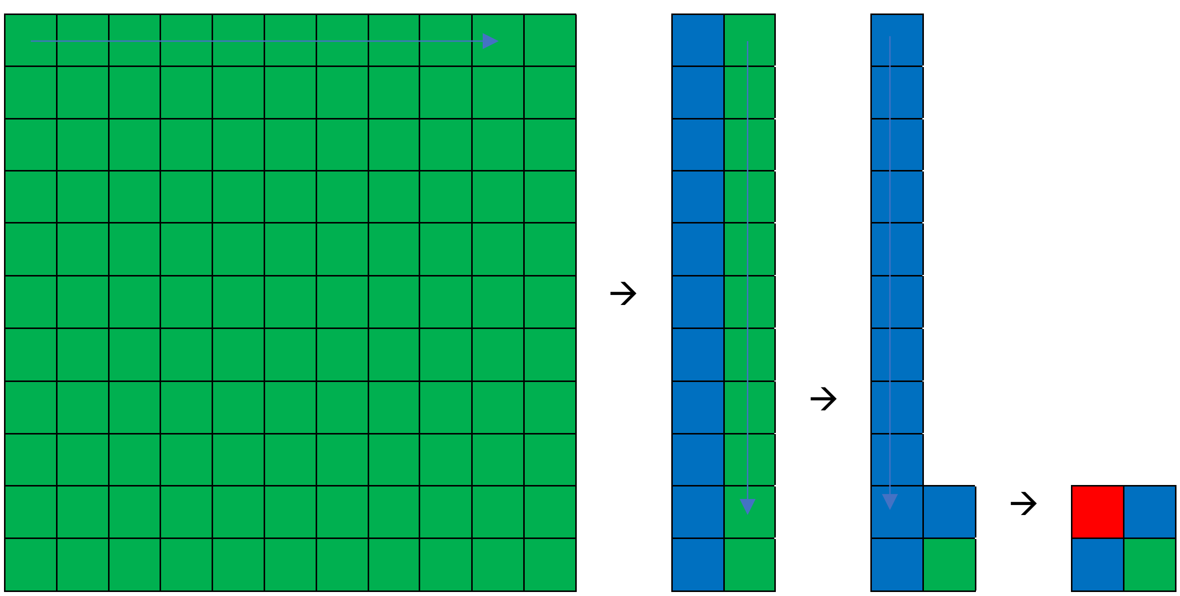
Prerequisites:

Constructing squares of binomials with the Golden Bead Material

Presentation One: Squaring Eleven, Twelve, and Thirteen

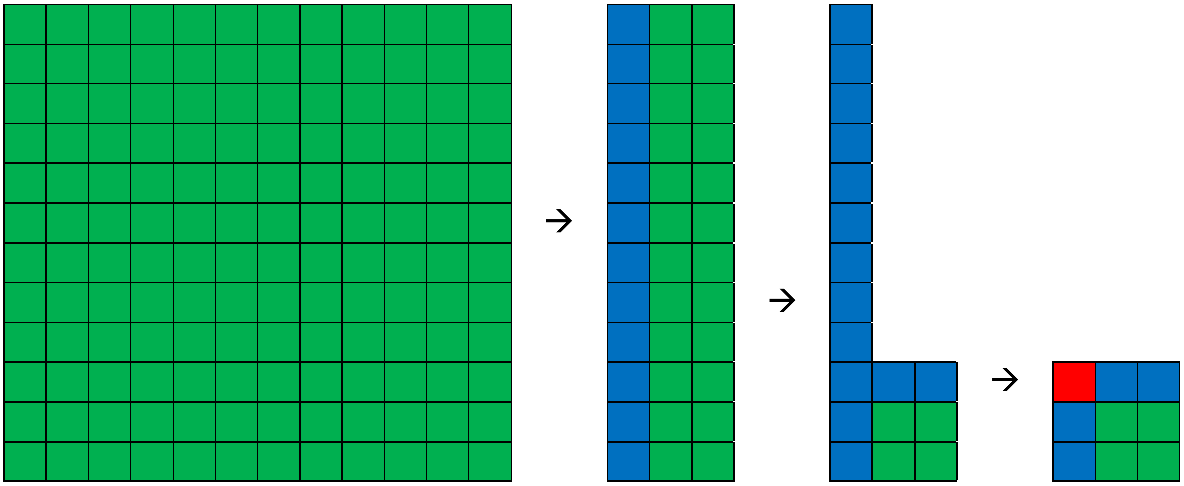
1. Demonstrate building the square of 2 using unit stamps. Build up to the square of 3. Ask for volunteers to build the square of 11 using only the green unit-stamps.
   This is an excellent time to go and get a cup of coffee.
2. Beginning in the upper left corner of the square, exchange ten unit-stamps in the top row for one ten-stamp. Repeat in each of the rows. **Make sure to count from the left**.
3. Now take ten one stamps from the right column, beginning at the top and replace them with a single ten-stamp.

   Beginning at the upper left, count ten ten-stamps and exchange for a single hundred-stamp. You should now have a square of only four stamps.



$11^2 = 121$

4. Ask the children to make the square of twelve on the mat using just ones stamps as before. Continue the process as before, following the same steps.



$12^2 = 144$

Follow-up

Children should construct and draw the 4 stages of squaring for the number 13.
If they wish, they can also build the square of any other teen. But it is going to take a LOT of unit stamps!

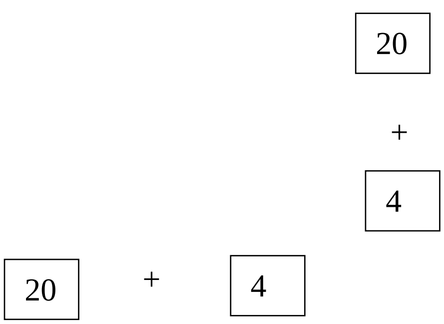
Control of Error

Calculator, Teacher

Presentation Two: Squaring Binomials with the Stamp Game

1. Remind the children of the previous lesson where 11, 12, and 13 were squared. Explain that these numbers are called **binomials** because they each are the sum of **two** numbers. Eleven is actually ten plus one, etc.

2. Explain that it is not necessary to go through the process of laying out all the ones stamps and replacing them with tens.
   Take the 20 and 4 numerals from each of the two numeral card sets and place as below with the addition signs.



3. Place the bottom row of stamps using two ten-stamps and four unit stamps.
   Explain that this represents 24 multiplied by 1; what we need is 24 multiplied by 24.
   Add another row above the previous row noting that you now have 24 multiplied by 2, or 48.
   Add two more rows and note that you now have 24 multiplied by 4, or 96.

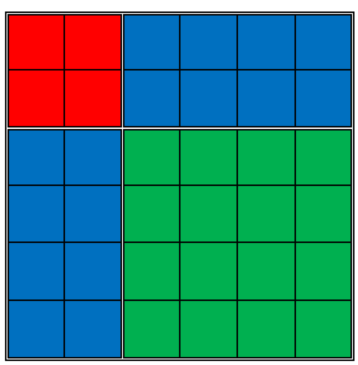
4. Note that you still need ten more rows of stamps to complete the problem. (UG!)

5. Remind the children that in the previous problems it was necessary to exchange when reaching combinations of ten. *Note: this reinforces number decomposition!*
   "What if instead of working harder, we can work smarter?" Show the children that it is easier to multiply to ten than to place all those rows out and then exchange. Show the children how each stamp of the next row can be increased by a factor of ten by using the stamp of the next place value. The result is the same as multiplying 24 by 10. The value of this row becomes 240.

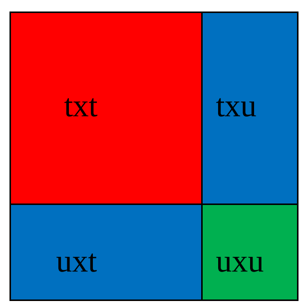
6. Ask if anyone can explain why the green area, the units, is a square and the red area, the hundreds, is a square. <When you square a number, you multiply it by itself. The green area, the units, is units x units, so it is the square of the units. The red hundreds area is tens x tens, so it is the square of the tens. The two blue areas are the cross-products: tens x units and units x tens. Most of the time, these will be rectangles because it is a simple multiplication of two different digits.>

7. Note that the two rectangles are identical except in orientation. (If desired, recollect the commutative law for multiplication that says that order doesn't matter.
   Tens x units = units x tens.

8. Show the Montessori Pattern Cards to emphasize the pattern. Stress that what the pattern card is showing is the pattern of rectangles and squares – relative size of the squares and rectangles does not match.



Stamp Game



Pattern Cards

9. Find the geometric/hierarchical value of each section of the square of 24. Record the following
   t x t – 4 hundred
   t x u – 8 tens
   u x t – 8 tens
   u x u – 16 units
   Find the numerical value of each section of the square and for the whole square:
   t x t – 1 hundred = 400
   t x u – 8 tens = 80
   u x t – 8 tens = 80
   u x u – 16 units = 16
   total 576

The square of 24 is 576: $24^2 = 576$
The 24 can be read along the bottom and the right side. The square, 576, is *rooted* on the base, 24.
Teacher-only note: this sequence for recording is far more beneficial than simply counting stamps to find the numerical value of the square. When children are ready to find square roots, the sequence of recording will be identical. We are sowing seeds here!

10. Using the Montessori Pattern Cards as a guide, have the children square 25 using stamps. Show them how to draw that figure on graph paper (1 stamp = 1 square), heavily outlining the 4 areas of interest (txt, txu, uxt, and uxu) and coloring in hierarchical colors.
    Ask them to record the numerical results next to the drawing of the square

NOTE: In this presentation, we methodically lay out the number being squared as a row as many times as necessary to represent multiplying by the units of the multiplier. We then lay out a row of stamps at ten times the value as many times as necessary to represent multiplying by the tens or the multiplier.
At some point in time, children will spontaneously transition from this methodical process to one where they lay out the root at the bottom and along the right side of the square, completing the square as a collection of internal squares and rectangles. Whenever this begins to happen, it is acceptable to modify presentations to follow the lead of the children.

Follow-up
Children should build, draw, and calculate the squares of a number of binomials.
Numerical recording should be <u>done by zone</u>, as demonstrated.

Control of Error
Calculator
Other students, teacher

Advanced Whole Number Operations: Cubing
Building Successive Numerical Squares and Cubes

Materials:

Bead Square and Cube for 1-9 *(10-square and cube are not needed)*
Completed Table of Perfect Cubes I from previous lesson
Wooden Cubing Material (irregular box)
Cubing Bars

Colored Counting Bars, offered by many manufactures, provides 20 bars of 1-10, which is more than needed, but is ideal for this and subsequent lessons. Shop around! Prices vary from about $50 to $400 at this writing!
Cuisenaire Rods or unit cubes from Cubing Materials can substitute for classrooms on a budget, but without the same visual impact – the colors are inconsistent and the lengths may not match precisely.
Combining the wooden materials with bead bars creates an impression but is very hard to manipulate

Direct Aim: to show the concrete elements needed when building from a cube to a successive cube, recording the numerical results

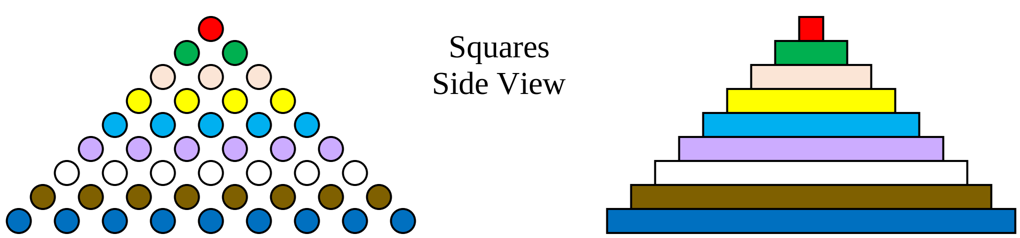
Indirect Aim: Preparation for the generalized equation for cubing a binomial
To concretely demonstrate the rate of change when successive numbers are squared or cubed.

Prerequisites:
Experience with squaring and cubing – concretely and numerically

Presentation:

1. Build the pyramid of squares 1-9 from the bead cabinet.

2. Remove the lid from the Wooden Cubing Material box and set it safely aside. Select one "square" of each color and build a pyramid of squares.



Bead Squares *Wooden Squares*

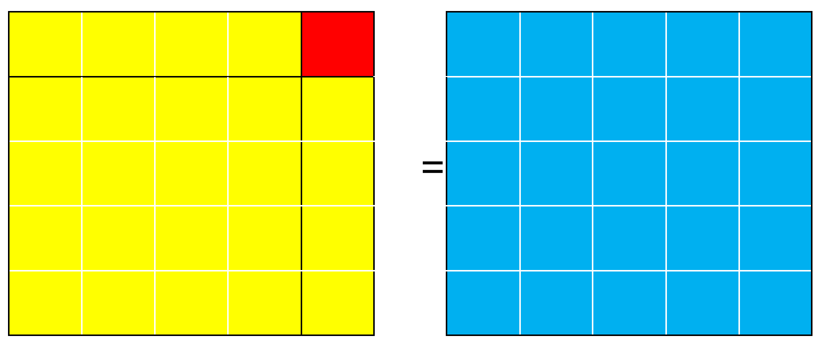
3. Invite the children to compare and contrast the two towers. Key comparison is the color correspondence – in each case *1* is red, *2* is green, etc. Key contrast is that with the bead squares, each quantity can be counted whereas the wooden squares represent quantity only with size and color – the wooden materials are more abstract.

4. Invite the children to build two towers of cubes: one with bead-cubes and one with wooden cubes. Once complete, again invite them to compare and contrast.

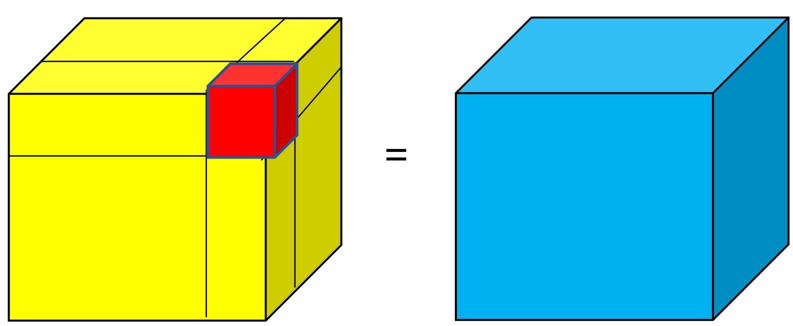
5. Set the bead materials aside. Explain that the rest of the lesson will use the wooden materials because they are easier to manipulate and because children don't need the support of being able to count the beads. Choose a square or cube at random and ask the children what it represents (3rd period question). If children hesitate, back up to second period questions like, "Who can show me the figure that represents 5^2? 2^3?
   Note: the "squares" are really prisms – they are a 3-dimensional representation of a 2-dimensional figure. The third dimension has been assigned to be 1 to create a 3-D object that can represent a 2-D shape without arithmetically affecting the product. This is why they represent squares.

6. Bring out the Colored Counting Bars. Ask what bead material they are equivalent to.
   <They are equivalent to bead bars: yellow counting bar is a 4-bar>

7. Review the process of building from a square to a subsequent square. Example: begin with 4^2 and build to 5^2.



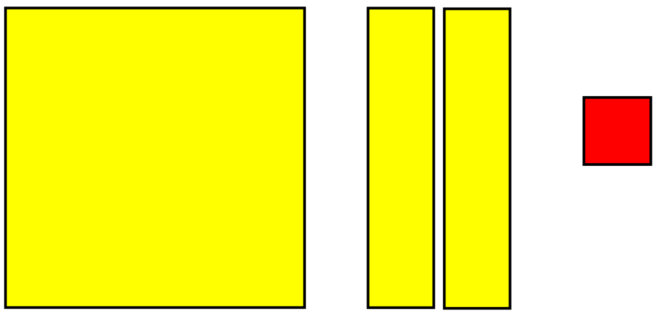
8. Children experiment with building from a cube to a subsequent cube.
   Example: begin with 4^3 and build to 5^3. After some experimenting, assist as needed.
   Add a 4-square to the front, side, and top.
   Add 4-bars on top of the squares on the side and the front. Add a third 4-bar on the rug at the intersection of the side and front squares.
   Add a 1-cube to the front top corner.
   Confirm that the constructed cube is dimensionally equal to the next larger cube.



9. Compare the rate of change when building from a square to a subsequent square to that when building from a cube to a subsequent cube. Children should note that when building from a cube to a subsequent cube the size of the figure gets a lot bigger a lot quicker than when building from a square to a subsequent square.

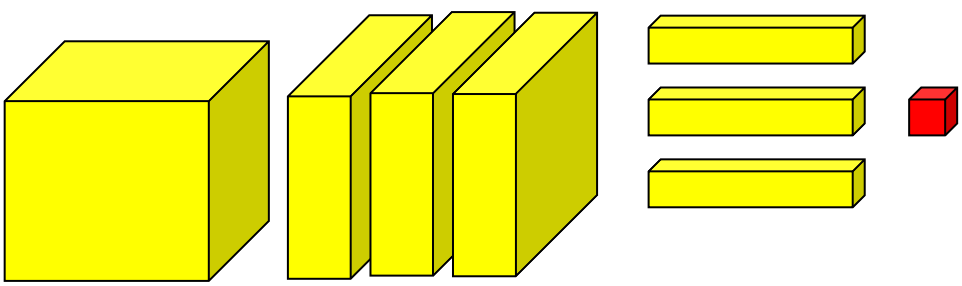
10. Look at the Table of Perfect Cubes and its graph. Ask children if the same seems to be true for all of the other numbers – that they get a lot bigger a lot faster when cubed than when squared. (If children completed extension activity – graphing – bring in that experience.) "Let's see why that might be true…"

11. Returning to the layout of square to the subsequent square, take the square that was constructed and decompose it into component parts, and write out each part numerically:



square of 4	+	2 (4-bars)	+	square of 1	= square of 5	
4^2	+	2(4)	+	1^2		
16	+	8	+	1	= 25 =	5^2

12. Repeat the process with the cube that was constructed. Use the chart of perfect cubes and charts as reference.



cube of 4	+	3 (4-squares)	+	3(4-bars)	+ unit cube	= cube of 5	
4^3	+	$3(4^2)$	+	3(4)	+ 1^3		
64	+	48	+	12	+ 1 = 125 =		5^3

13. Invite comparison between the two. Key observation is that when going from a square to a subsequent square, one builds on 2 sides; when going from a cube to a subsequent cube, one builds on 3 sides. That is one reason why the numbers get so much bigger so much faster.

14. Return all materials to the boxes.

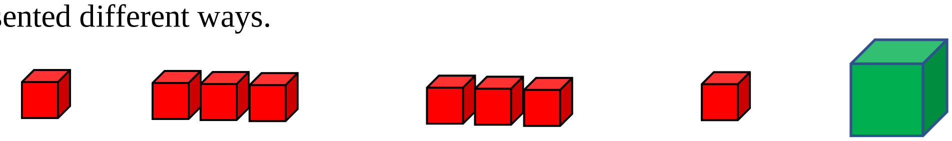
15. Invite the children to make 3 groups. Each group builds from a cube to a subsequent cube. Choices are $2^3 \rightarrow 3^3$ or $6^3 \rightarrow 7^3$ or $8^3 \rightarrow 9^3$
    (Avoid building from $1^3 \rightarrow 2^3$ because it is impossible to tell 1-squares from 1-bars from 1-cubes once they are removed from the cube.)
    Each team should build onto the smaller cube to construct a cube that is dimensionally equivalent to the larger cube.
    Once they have built the cube, they decompose it and write out the numerical values of each of the components, confirming through arithmetic that the cube that they constructed is equivalent to the next larger cube. Use the Table of Perfect Squares and Cubes to confirm.

Follow-up: Children should individually and as independently as possible choose a pair of adjacent cubes and build from one to the next, decomposing the constructed cube and writing out the values of the component parts numerically as was demonstrated in the lesson. Confirm the resulting value with the Table of Perfect Squares and Cubes.
Once successful with the above, they may build from $9^3 \rightarrow 10^3$. There is no physical 10^3 present in the box to confirm that they have indeed built 10^3,. However, when children decompose the cube and into its component parts and writing out the values numerically, they know from prior experience in the decimal system that 10^3 is 1000. Their components should sum to 1000.

Control of Error: Table of Perfect Squares and Cubes I

Extension: <u>The most flexible thinkers</u> will find it fun and interesting to build from $1^3 \rightarrow 2^3$.

This is challenging because this material creates a physical representation of squares 3-dimensionally by assigning the height of the square to 1 unit. In actually, if it were a true square, the height would be 1 atom. So, the true dimensions of 4^2 in this material is 4 x 4 x 1. Numerically, this is 16.
This all works well until considering the square of 1. Dimensionally it is 1 x 1 x 1. It looks like 1^3 physically, even though it represents 1^2. Similarly, a 1-bar is dimensionally 1 x 1 x 1, looking just like 1^2 and just like 1^3 even though it represents 1^1
If they have internalized the pattern, they will find it entertaining to see the same cube represented different ways.



1-cubed	+	3(1-squares)	+	3(1-bars)	+	1-cubed		
1^3	+	$3(1^2)$	+	3(1)	+	1^3		
1	+	3	+	3	+	1	= 8	$= 2^3$