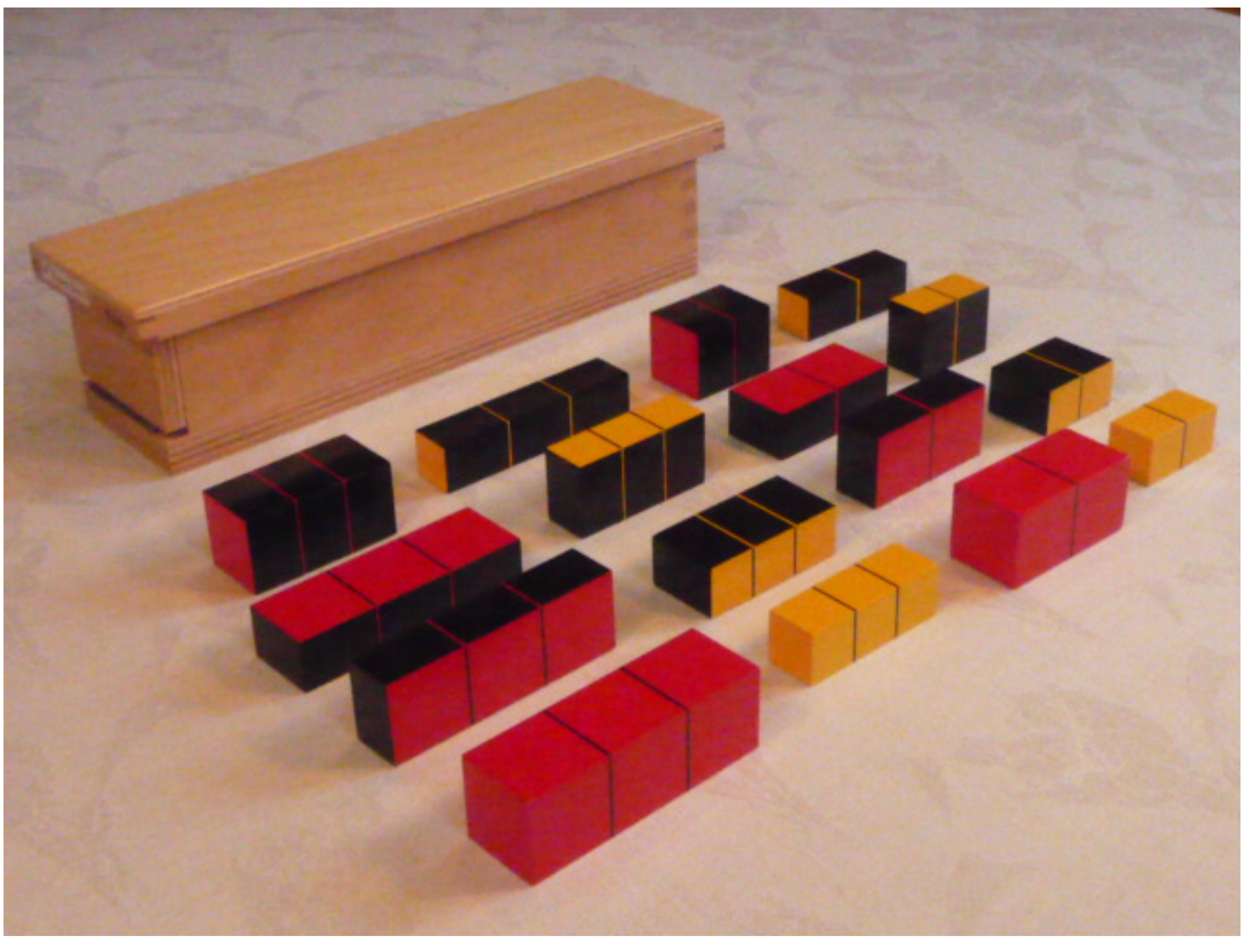
Explanation of the derivation and design of the material for

The Fourth and Fifth Powers of a Binomial

from ***Psicoaritmetica***

by Maria Montessori

translated and interpreted by Betsy Lockhart



In the summer of 2010, a Montessori Education Center of the Rockies graduate contacted us to ask about some unfamiliar materials sold by Nienhuis that had been stored at her school. They were clearly upper level math materials, labeled as various versions of the fourth and fifth power of the binomial, but were not anything that she had seen before. The prospect of representing expressions beyond the third dimension with concrete materials was, to say the least, intriguing! How do we go beyond length, width and height in a concrete fashion? I had to know how this could be done. This began a lengthy and interesting investigation of these materials.

Inquiries revealed no colleagues with experience on these materials. Fortunately, the training center has a copy of the now out-of-print *Pscioarithmetica* by Maria Montessori, originally written in 1934 but reprinted in Italian in 1971, which describes and illustrates these materials. (Montessori, Maria *Psicoarithmetica* Milan: Aldo Farzanti Editore, 1971.) Unfortunately, I do not read Italian. Fortunately, there are on-line translation resources where one types in a passage and the website translates on a sentence-by-sentence basis. (I used Prompt Online: http://translation2.paralink.com/.) Unfortunately, only a few sentences can be entered at a time, which can be quite tedious, and the resulting "paragraph" is often less than elegantly expressed, requiring the reader to interpret Dr. Montessori's writing by altering the syntax of the sentence or substituting a series of words with a word or phrase more familiar to the modern American ear, with a solid knowledge of mathematics and algebra. Fortunately, these are passions of mine, so I found the process compelling. This paper is the result of that investigation.
The following document was produced by translating using Prompt Online and then interpreting the text from Maria Montessori's book *Psicoaritmetica*. Passages in quotes are near-direct translations from Dr. Montessori. Other passages draw upon the author's knowledge of algebra and the use of other Montessori materials.

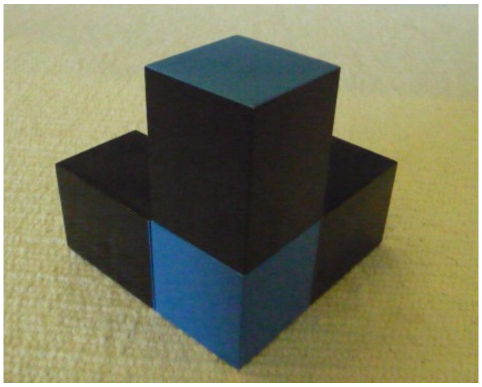
Note: Throughout this discussion, terms describing concrete materials are **bold**, while purely algebraic expressions are *italicized*. Terms that apply equally to the algebraic and concrete representation of the expression are in normal font.

The Concrete Representation of $(a+b)^4$

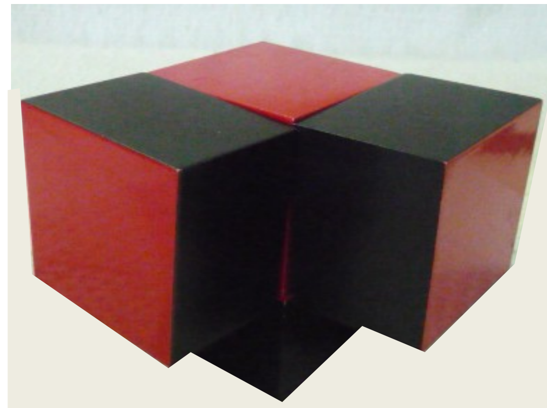
Following Dr. Montessori's lead, before considering $(a+b)^4$, let us re-acquaint ourselves with $(a+b)^3$ and its concrete representation: the binomial cube material. Consider the algebraic expansion of $(a+b)^3$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

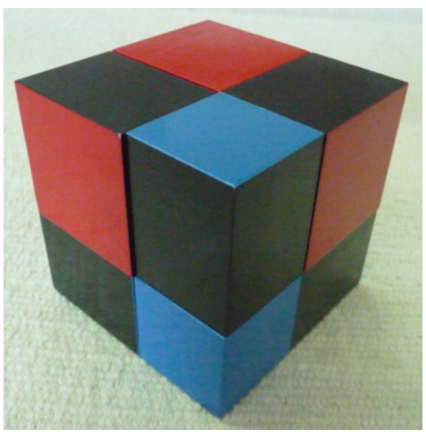
In order to manufacture the material, Dr. Montessori assigned dimensions to *a* and *b*: a = 4cm and b = 3 cm. Faces representing a^2 are painted red, while those representing b^2 are painted red. Faces that represent the cross-product, ab, are black.



base = $b^3 + 3b^2a$



top = $a^3 + 3a^2b$



$(a+b)^3$= base + top =

$a^3 + 3a^2b + 3b^2a + b^3(a+b)^3$

Note: orientation of the binomial cube is consistent with *Psicoaritmetica*, but up-side down the to Nienhuis catalog.

Algebraic binomial cube

geometrically equivalent to the binomial cube, colors consistent with $(a+b)^4$ material.

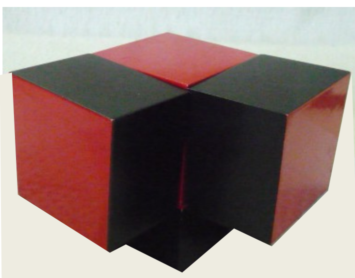
The difficulty that arises when we raise (a+b) to the fourth power is that we cannot physically represent the fourth dimension. Scientifically, the fourth dimension has been considered to be "time", which is useful in studying the concept of Relativity. Mathematically, the fourth dimension is sometimes treated as a rotational movement of a 3-dimensional figure. Either way, the fourth dimension cannot be represented by a static set of materials. In *Psicoaritmetica*, Dr. Montessori poses the question, "If the bodies cannot extend in the physical realm beyond the 3rd dimension, how can we represent the formula in which not one of the terms can be represented physically?" (Montessori, 367)

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

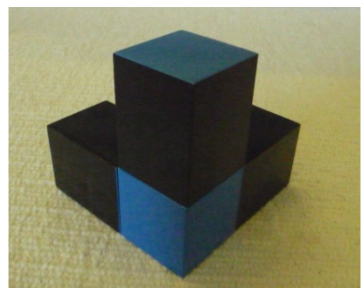
Note that each term in the expansion takes us into the fourth dimension; not even one term can be represented in 3-dimensional space. Dr. Montessori noted, however, that "…the terms can be rearranged (penetrated) in order to distinguish quantities that have common factors that indicate a multiplicity of objects. If we can isolate a common factor, it simply indicates the number of times the multiplicand should be repeated. To do this, analytically follow the transition from the third to the fourth power… The binomial cube – in its entirety – can be repeated several times (a+b times)." (Montessori, 368)

$$(a+b)^4 = (a+b)^3(a+b)$$

Returning to the binomial cube – the first term in the expansion of $(a+b)^4$, we realize that "(the) formula for the expansion of $(a+b)^3$ consists of two parts: two halves of the cube, each of which consist of 3 square-based rectangular prisms resting on three adjacent faces of a perfect cube.



$a^3 + 3a^2b$ (top)



$b^3 + 3b^2a$ (bottom)

"Now proceed to the multiplication of the cube by one repetition of (a+b) without distributing or adding them (which results in the following)

$$(a+b)^4 = (a^3+3a^2b + b^3 + 3b^2a)\ (a+b)$$

(top + bottom) (number of repetitions)

"Now separate the two parts by layers, geometrically:" (Montessori, 369)

$$(a^3+3a^2b)a + (b^3 + 3b^2a)a \rightarrow \quad \text{top} \cdot a + \text{bottom} \cdot a$$

$$+\ \underline{(a^3+3a^2b)b + (b^3 + 3b^2a)b} \rightarrow \quad +\ \text{top} \cdot b + \text{bottom} \cdot b$$

"While we might combine like terms in algebra, physically this has no meaning, because it takes us into the 4th dimension." For our own edification, we can do the algebraic distribution just to be sure that we still have an equivalent expression:

First term: $(a^3+3a^2b)\,\mathbf{a} = a^4+3a^3b$

Second term: $(b^3 + 3b^2a)\mathbf{a} = b^3a + 3b^2a^2$

Third term: $(a^3+3a^2b\,)\mathbf{b} = a^3b+3a^2b^2$

Fourth term: $(b^3 + 3b^2a)\mathbf{b} = \underline{b^4 + 3b^3a}$

$a^4+4a^3b+6a^2b^2+4ab^3+b^4$ (equivalency confirmed)

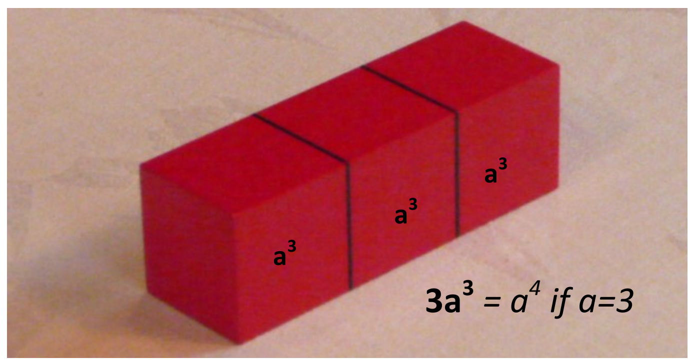
Recall that, in order to manufacture the material to illustrate $(a+b)^3$, Dr. Montessori assigned physical dimensions to *a* and *b*: 3cm and 2cm respectively. In the case of illustrating the binomial raised to the fourth power, Dr. Montessori preserved the assignation of <u>physical dimensions</u> for the <u>cubic expansion</u> of the binomial, but further assigned <u>abstract numerals</u> for the <u>multiplier</u> (3 and 2, respectively). This distinction will be preserved throughout the derivation in order to maintain the concrete aspect of the cubic expansion of the binomial. Ascribing abstract numerals for the multiplier creates a specific case, but once the specific case is illustrated, we can extend the concept by analogy.

First term: $\boldsymbol{(a^3+3a^2b)a}$ = $3(a^3+3a^2b)$ when a=3

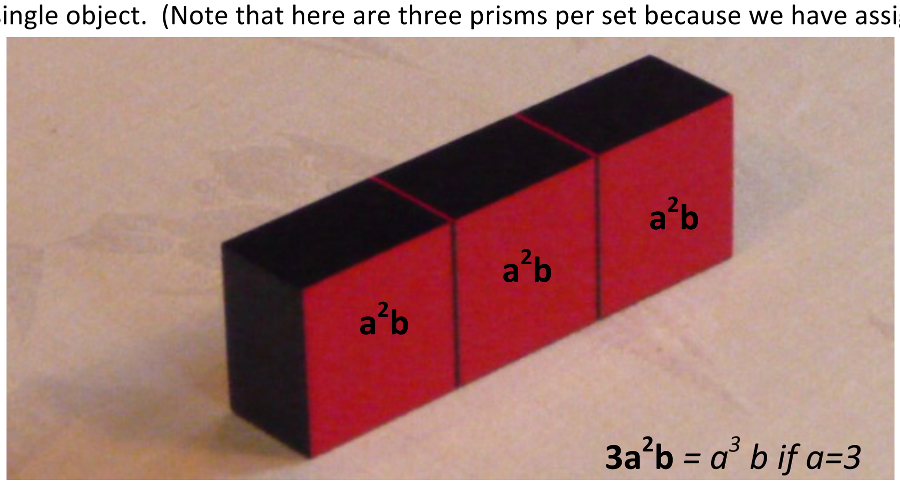
"We start from the first part and assign numerical values to *a* and *b* calling on arithmetic to help in the construction, supported by algebra and geometry." (Montessori, 370) We use the numerical values of a=3 and b=2 <u>for the multiplier only</u>, to indicate the number of repetitions of the given volume are needed to create the fourth power of the binomial, thus:

$$
\begin{aligned}
(a^3+3a^2b)a = (a^3+3a^2b)3 = {} & (a^3+3a^2b) \\
& + (a^3+3a^2b) \\
& \underline{+ (a^3+3a^2b} \\
& \;\; 3a^3+3(3a^2b)
\end{aligned}
$$

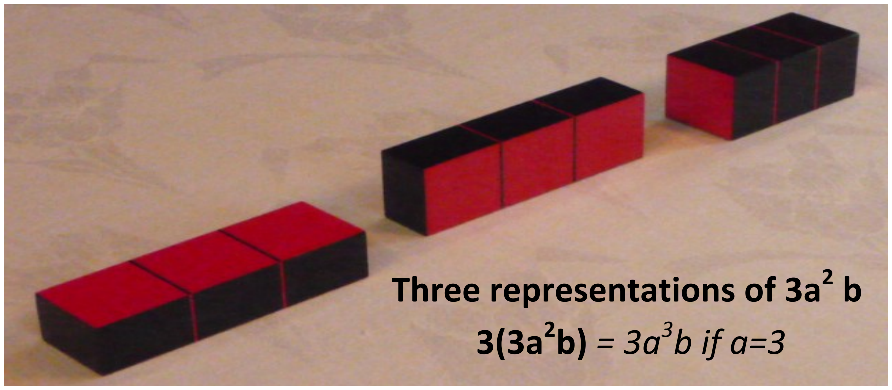
The first term, therefore, has two parts.
The first part consists of 3 a-cubes.
"if we merge the cubes into a single object we get a square based rectangular prism corresponding to the term a^4."
(Montessori, 371)



The second part of the first term consists of three objects, each of which is comprised of three rectangular prisms. "Now to the three prisms represented by $3a^2b$. If we construct these using the same dimensional values, they will each be 3 x 3 x 2."[5] We then merge the three prisms together to form a single object. (Note that here are three prisms per set because we have assigned a=3.)



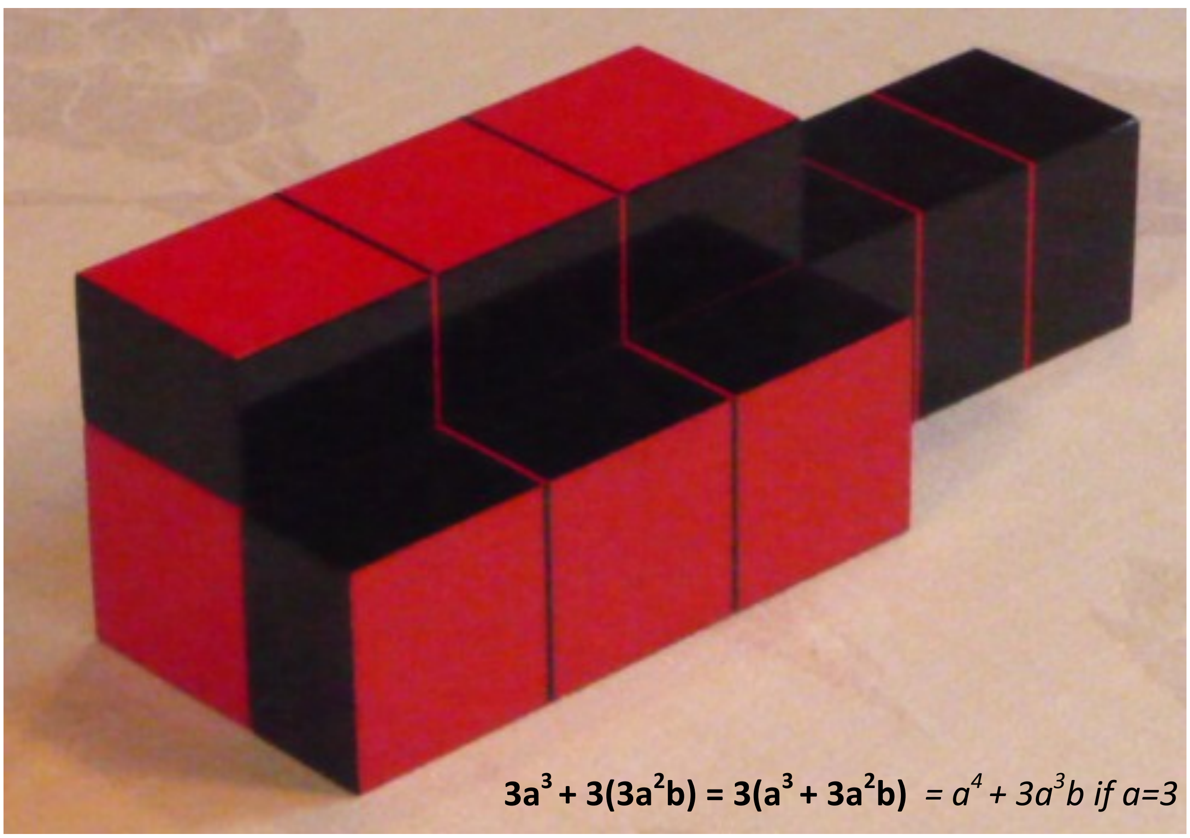
We have three repetitions of this volume in this expansion, which concretely express the three axes of the expansion: length, width and height.



Three representations of $3a^2b$

$\mathbf{3(3a^2b)}$ *$= 3a^3b$ if $a=3$*

Now we complete the first term by placing all of the pieces correctly in three-dimensional space. "We place them relationally to the a^4 cube as we did to a^3 in $(a + b)^3$." (Montessori, 373)



$\mathbf{3a^3 + 3(3a^2b) = 3(a^3 + 3a^2b)}$ *$= a^4 + 3a^3b$ if $a=3$*

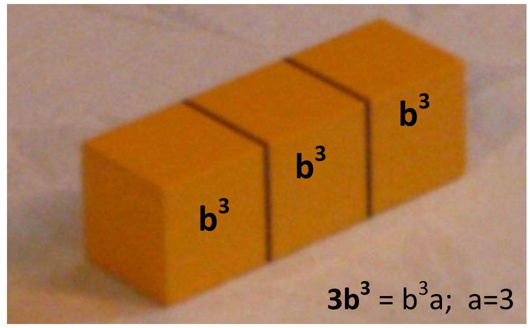
"Although this does not represent the <u>general</u> case, we clearly demonstrate the possibility of the fourth power of any combination. " That is to say, we could assign *a* to any positive integer and obtain an analogous result.

* *

Second term: $(b^3 + 3b^2a)a$ = $3(b^3 + 3b^2a)$ when a=3

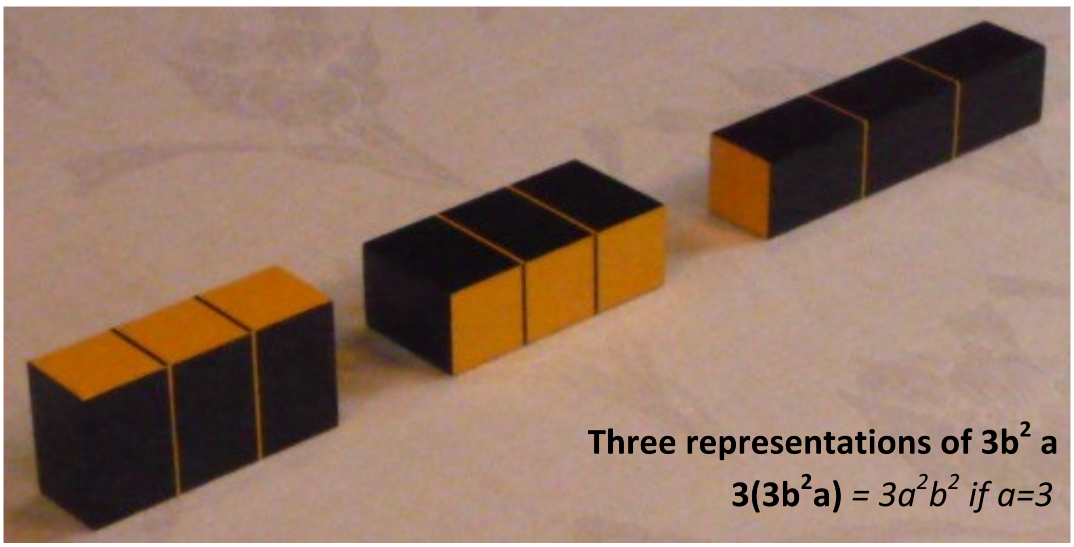
In like fashion, the second term has 2 parts.

$$(b^3+3a^2b)a = 3(b^3+3a^2b) = (b^3+3a^2b)$$
$$+ (b^3+3a^2b)$$
$$\underline{+ (b^3+3a^2b)}$$
$$3b^3+3(3a^2b)$$

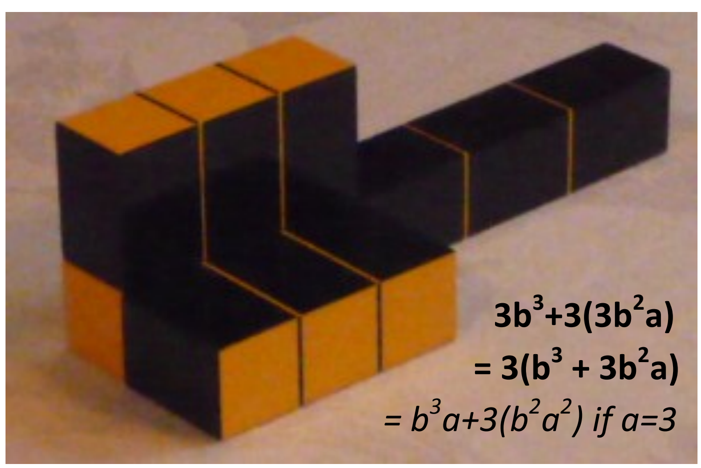


The first part is $3b^3$or the cube of b, repeated 3 times if a=3.
Once again, we merge the 3 cubes into a single element.

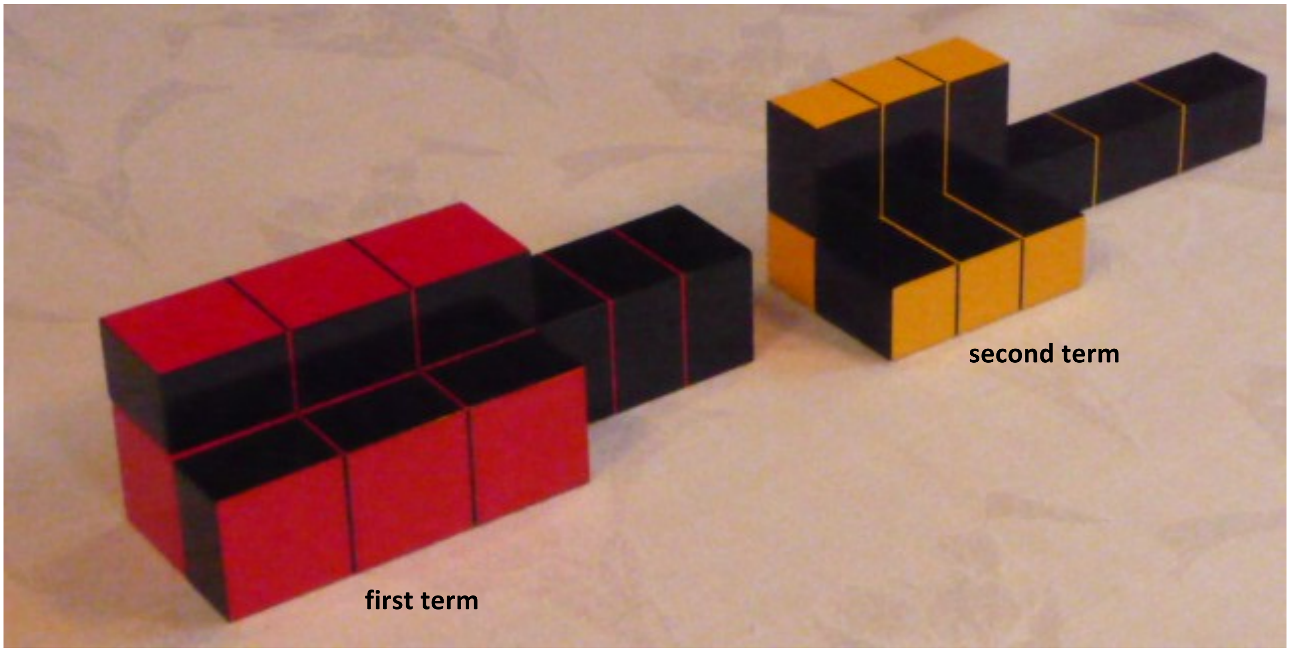
The second part is represented by three objects that are each created by merging three prisms that are dimensionally 2 x 2 x 3.



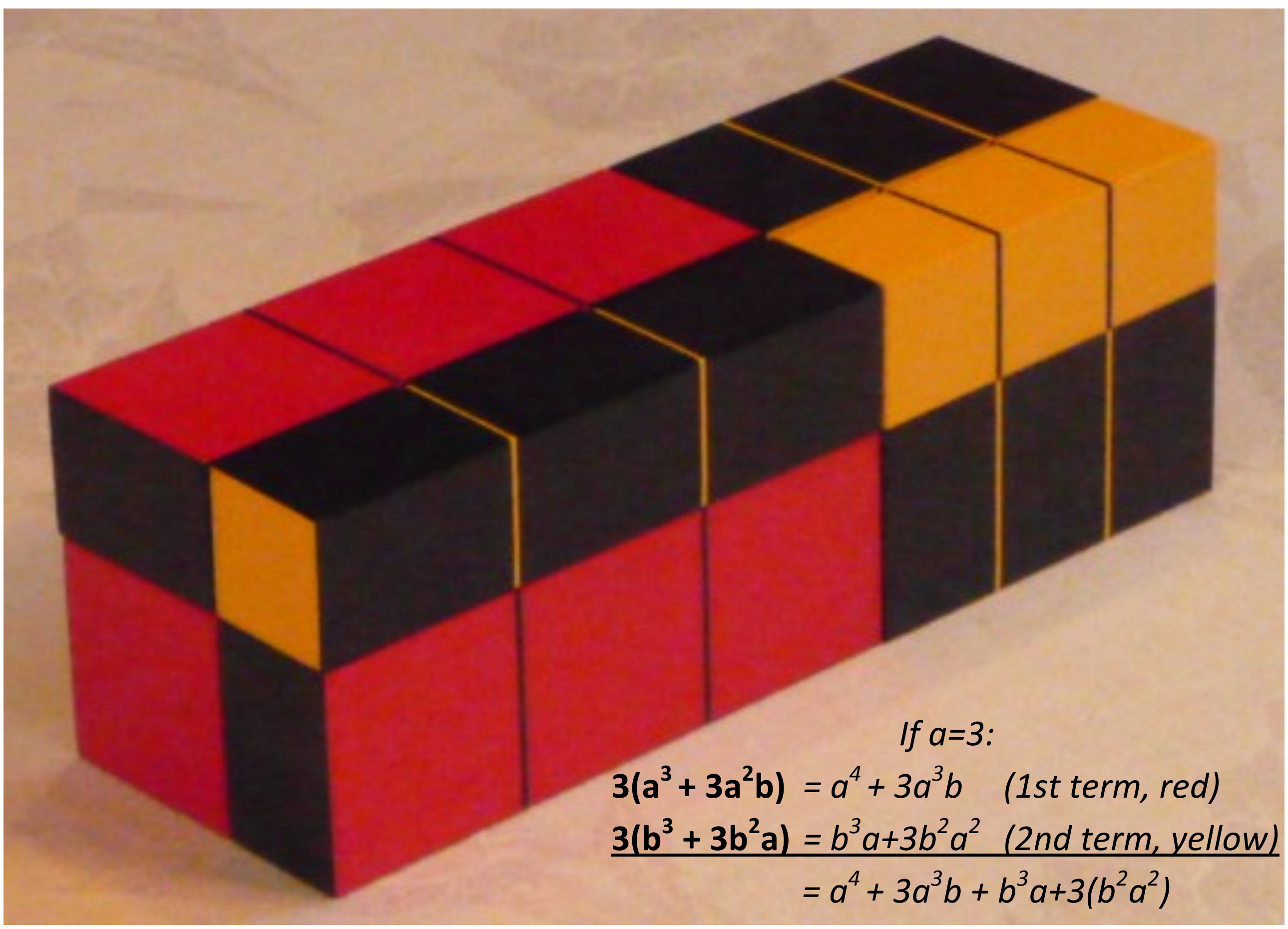
When the 3 cubes and the three sets of rectangular prisms are combined, the result is the second term



Combining the first and second terms



If one could rotate the entire assemblage of the second term such that the assembled object is resting on one of the sets of rectangular prisms with the set of cubes on the top face closest to the observer, it could then be nested with the first term, to produce a large rectangular prism as pictured below.

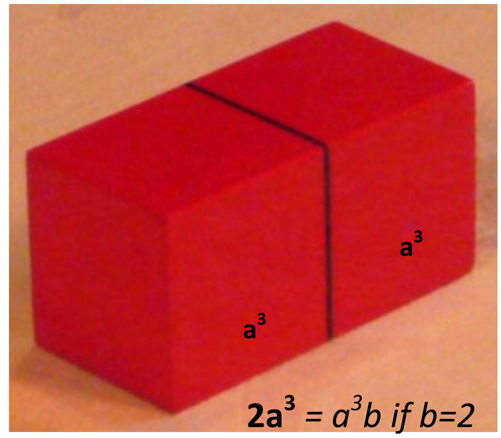


Third term: $\boldsymbol{(a^3+3a^2b)b}$ $= 2(a^3+3a^2b)$ when $b=2$

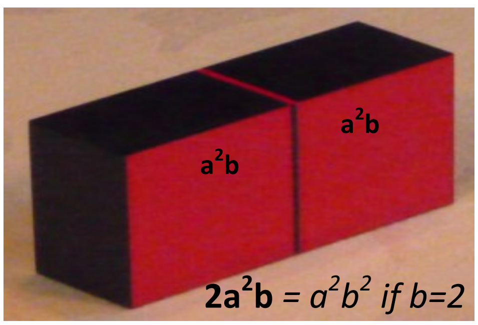
Once again, we stop short of distributing b or its value, 2, algebraically. Rather, we look to the assignation of b=2 determine the number of repetitions to make of each element in the object. Thus:

$$(a^3+3a^2b)b \;=\; 2(a^3+3a^2b) = (a^3+3a^2b)$$
$$\underline{+\;(a^3+3a^2b)}$$
$$2a^3+2(3a^2b)$$

As with the previous two terms, we break this term into two parts. The first part, $2a^3$, is shown to the right It is the cube of *a*, repeated twice. We merge the two cubes into a single element, as before. Thus, "the cube of 3, repeated 2 times is represented as a rectangular prism whose base is (two squares, each) equal to a^2." (Montessori, 374)



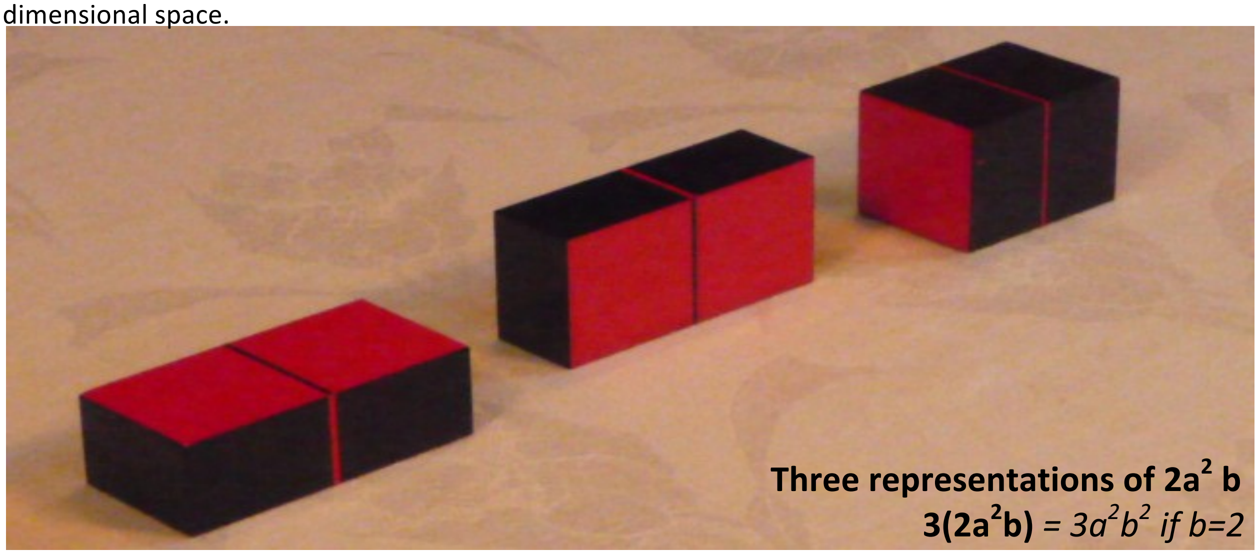
$\mathbf{2a^3}$ $= a^3b$ *if* $b=2$



$\mathbf{2a^2b}$ $= a^2b^2$ *if* $b=2$

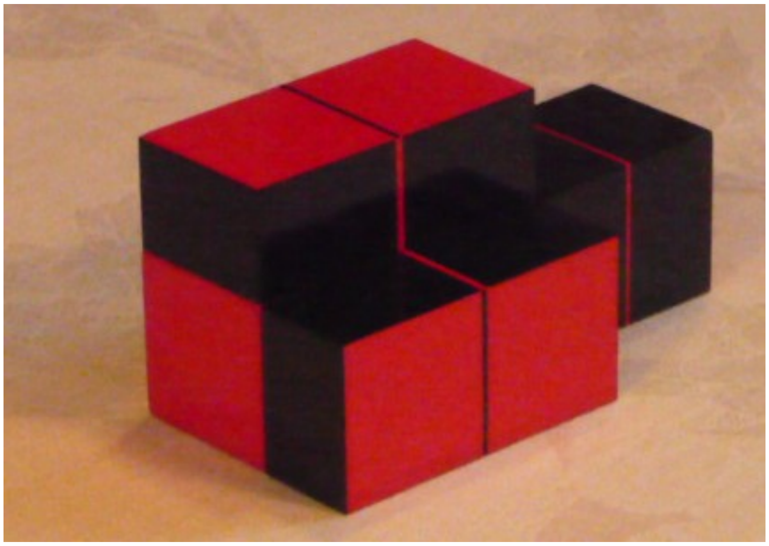
The second part is the created by merging the two prisms that are each dimensionally a x a x b (or 3 x 3 x 2) into a single element.

When we view the three representations of $2a^2b$ below, we are able to more clearly distinguish between repetitions due to the assignation of numerical values to a and b and repetitions due to the algebraic expansion of the binomial into three dimensions: length, width and height. Here, there are two repetitions merged into each object because b=2, and three objects because of the binomial expansion into 3-dimensional space.



Three representations of $2a^2$ b
$\mathbf{3(2a^2b)}$ $= 3a^2b^2$ *if* $b=2$

Combining the two parts of the third term, we have the following volume:



$2a^3 + 2(3a^2b) = 2(a^3 + 3a^2b)$ *$= a^3b + 3a^2b^2$ if b=2*

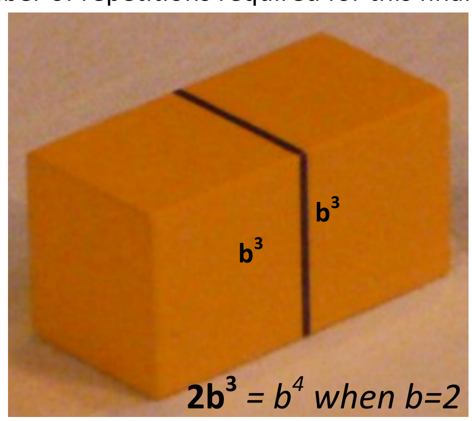
* * * * * * * * * ** *

***Fourth term:* $(b^3+3b^2a)b$** = $2(b^3+3b^2a)$ when b=2

"We reached the last part of the formula: $(b^3+3b^2a)b = b4 + 3b^3a$." (Montessori, 375) Once again, we stop short of distributing b or its value, 2. It represents the number of repetitions required for this final term.

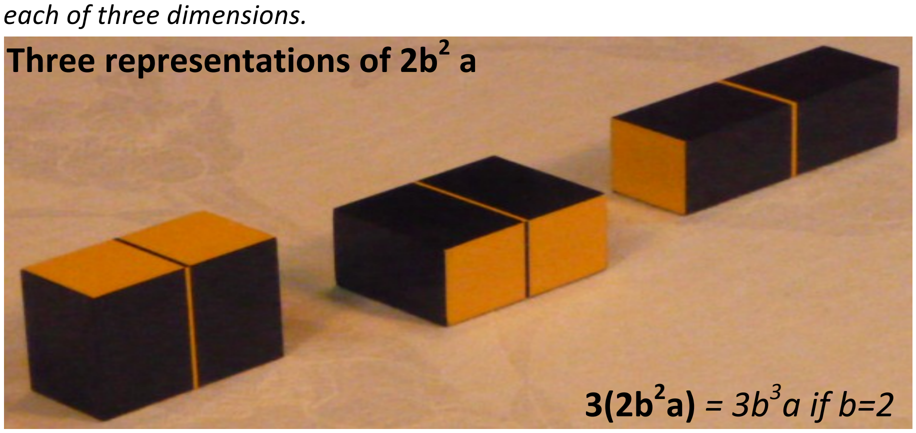
$$(b^3 + 3b^2a)b = 2(b^3 + 3b^2a) = (b^3 + 3b^2a)$$
$$\underline{+\ b^3 + 3b^2a}$$
$$2b^3+2(3b^2a)$$

As previously, we break this term into two parts. The first part, $2b^3$, is shown to the right. It is the cube of b, repeated twice. We merge the two cubes into a single element, as before.



$2b^3$ *$= b^4$ when b=2*

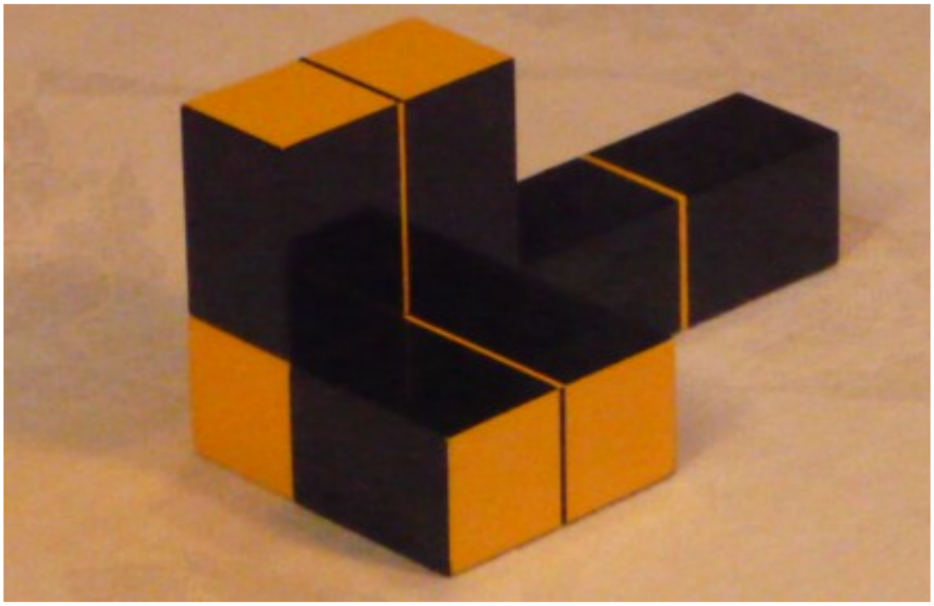
The second part is the created by merging the two prisms that are dimensionally a x b x b into a single element. *Note: there are still 3 representations of this square-based rectangular prism, one oriented to each of three dimensions.*


Three representations of $2b^2$ a

$3(2b^2a)$ *$= 3b^3a$ if b=2*

Fourth term: $(b^3+3b^2a)b$ (continued)

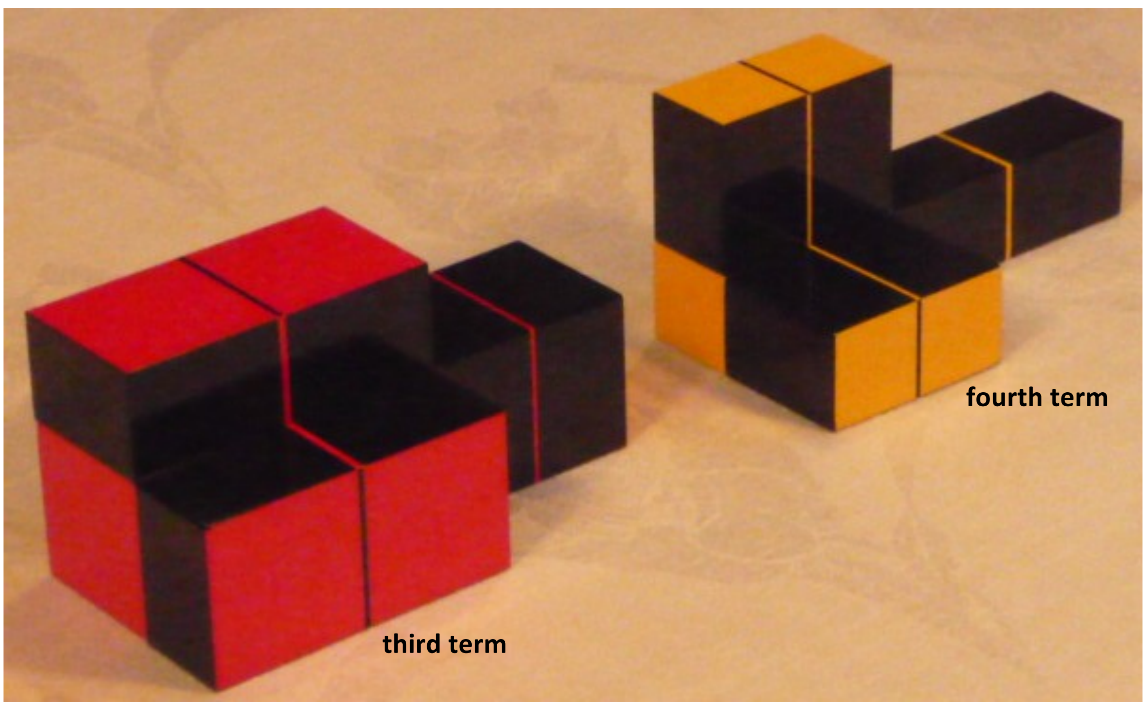
Physically, combining the two parts of the fourth term, we have the following volume:



$$\mathbf{2b^3 + 3(2b^2a) = 2(b^3+3b^2a)} = b^4 + 3b^3a \text{ when } b=2$$

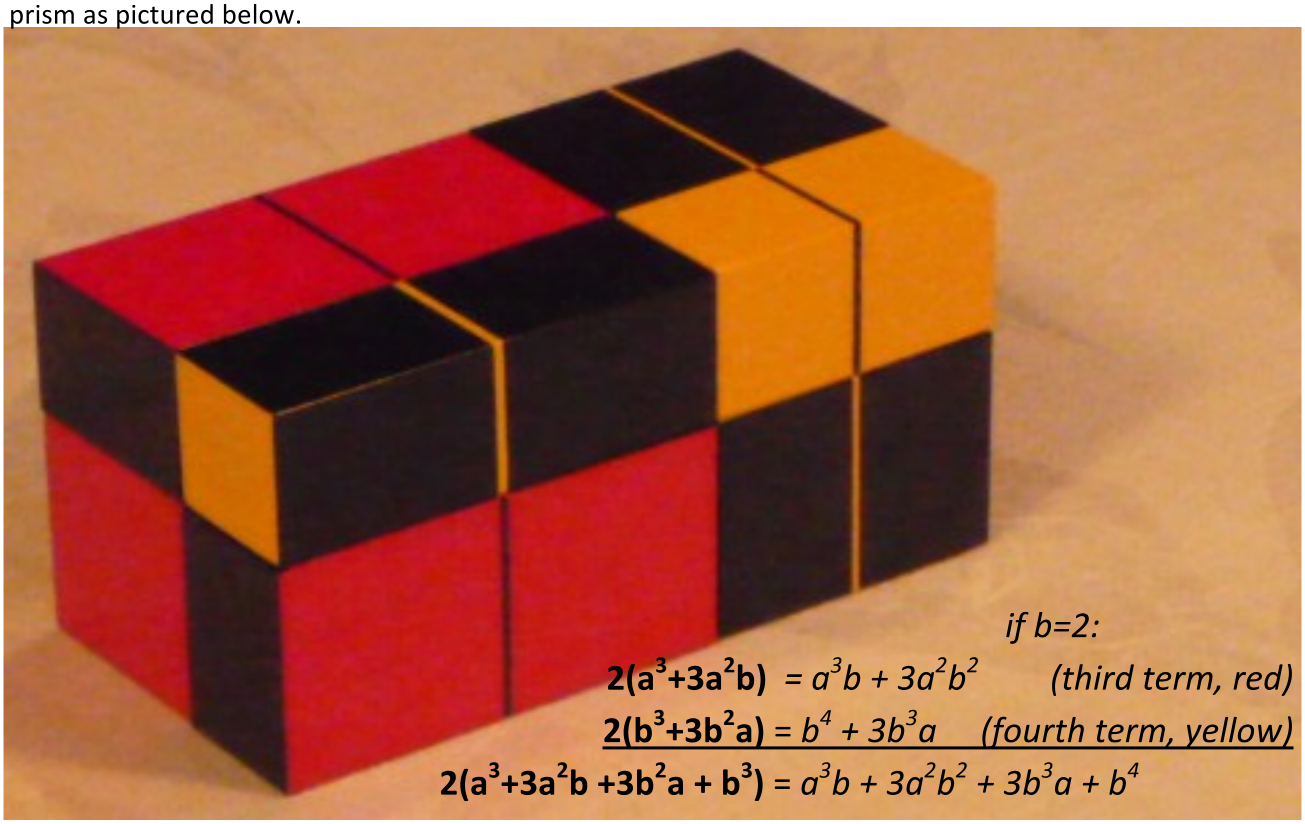
* * * * * * * ** *

Combining the third and fourth terms



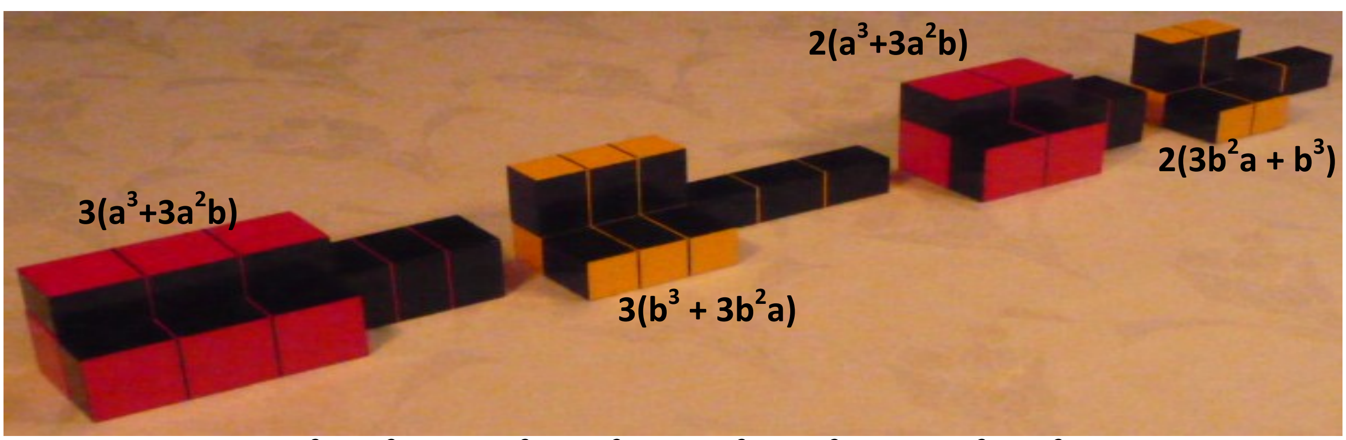
Combining the third and fourth terms (continued)

Combining the third and fourth terms as before, by rotation and nesting, produces a large rectangular prism as pictured below.



* *

Combining all four terms



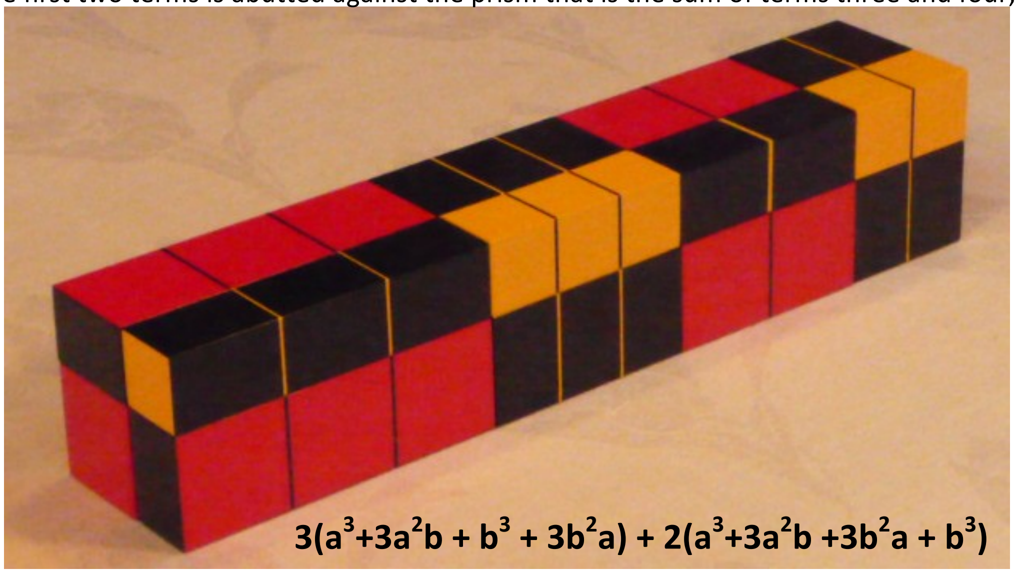
$3(a^3+ 3a^2b) + 3(b^3 + 3b^2a) + 2(a^3 + 3a^2b) + 2(3b^2a +b^3)$

*Thus, if **a=3** and **b=2**, the above can be represented by*

$a^4+ 3a^3b + b^3a + 3a^2b^2 + a^3b + 3a^2b^2 + b^4 +3b^3a$

Combining all four terms (continued)

When, at last, all four terms are brought together to represent the fourth power of the binomial, the prism that is the sum of the first two terms is abutted against the prism that is the sum of terms three and four, thus:

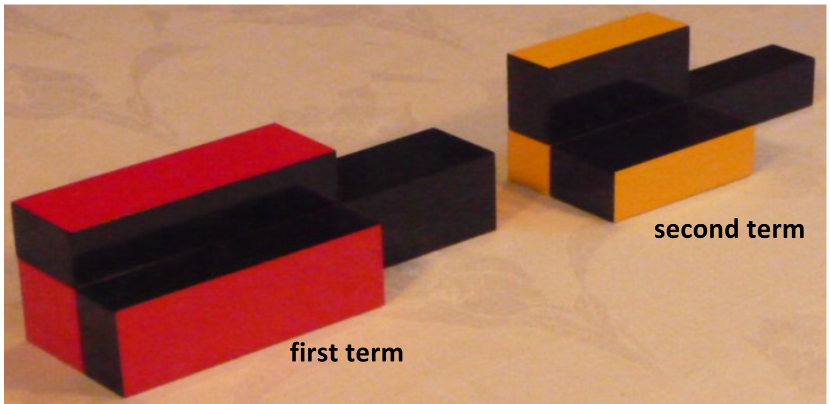


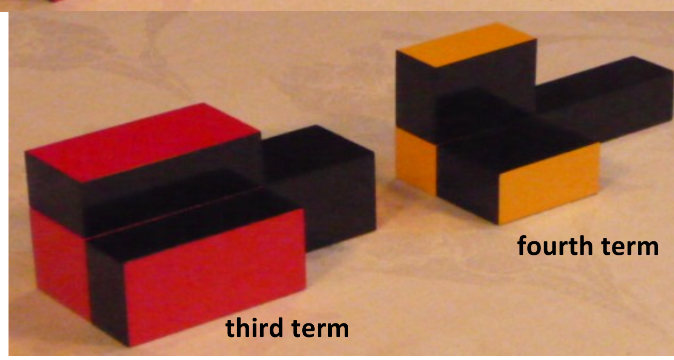
Note: The placement of elements within the prism, above, is consistent with that in *Psicoaritmetica*. The Nienhuis catalog shows the outer layer of the second and third terms reversed, such that the yellow elements are contiguous and the black elements are contiguous.

And so the identity is confirmed for the specific case where a=3 and b=2. One might choose to repeat this exercise with a variety of different values for *a* and *b*, each time confirming the algebraic identity. Thus, the materials can be used to represent the general case. "With these pieces can be a rectangular prism that represents the form taken by the fourth power of our duo. Given the measures that we (assigned), this prism will have a square section of 5 x 5 cm, and be 25cm in length." (Montessori, 376)

$(a+b)^4$, Box Two

The second through fourth boxes for the fourth power of the binomial (a+b) are not described in the 1971 edition of *Psicoarimetica*. However, an examination of the materials yields a few observations. Box Two is nearly identical in content to Box One. In the second box, the lines delineating the number of repetitions contained in each object (3 or 2) are removed, leaving only solid-colored prisms for each object. Assembling the four terms as with the previous box yields the identical geometric relationships as solid elements:

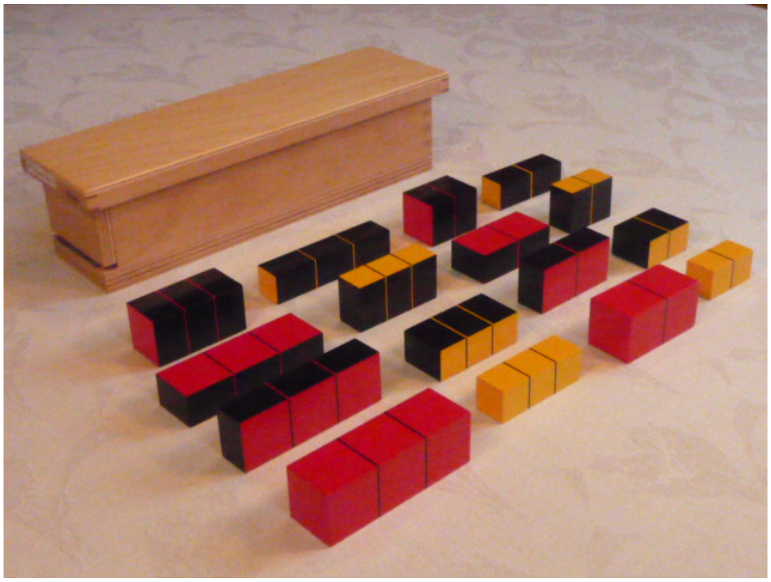




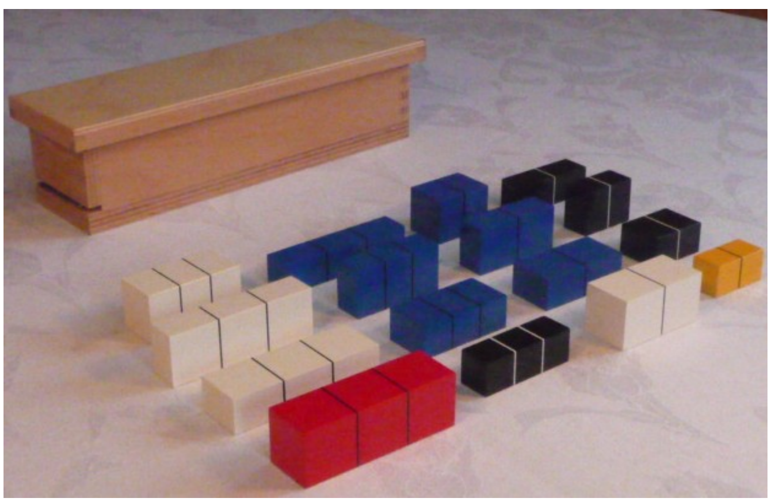
Combining the four terms as before similarly yields a single rectangular prism:

$(a+b)^4$, Box Three

Boxes One and Two have a color scheme that matches that of the arithmetic cube of the binomial. That is, objects related to a ($3a^3$, $2a^3$, or a prism with faces based on a^2) are red. Objects related to b ($3b^3$, $2b^3$, or a prism with faces based on a^2) are yellow. The final assembled prism allows us to see in the completed prism the component parts from which it was built.

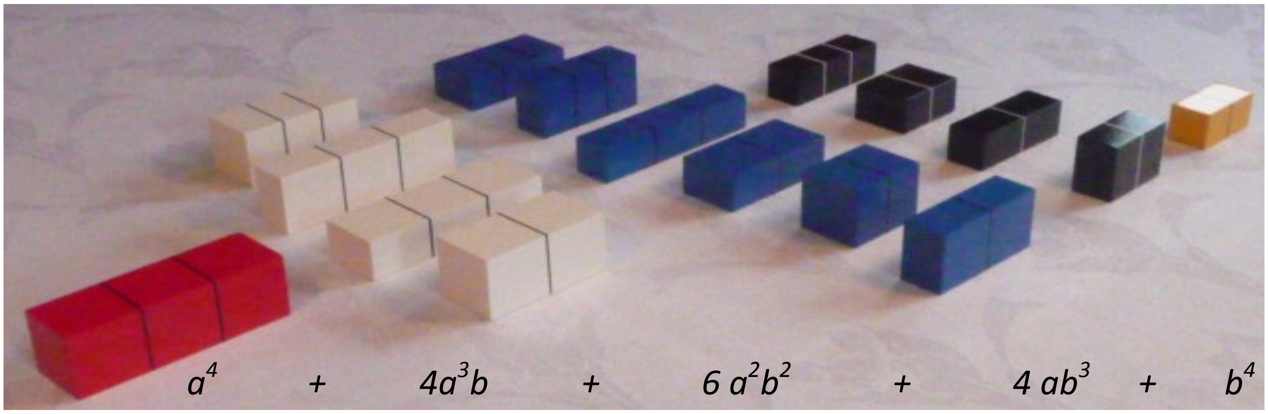


If we wish to examine the geometry of the terms that make up the algebraic expansion, we are better served by color coding the component prisms according to the term in the expansion. This is what has been done in Boxes Three and Four.



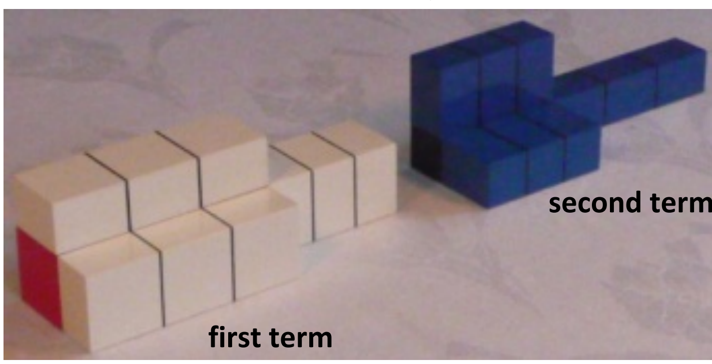
When one compares the various objects from Box Three to Box One, it is easy to see the volumetric correlation. The only difference between the contents of the two boxes is the colors chosen for the various objects. Recall that there are 5 terms in the expansion: $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. The colors in Boxes Three and Four are done so according to the terms in the expansion. We explore that aspect of the material next.

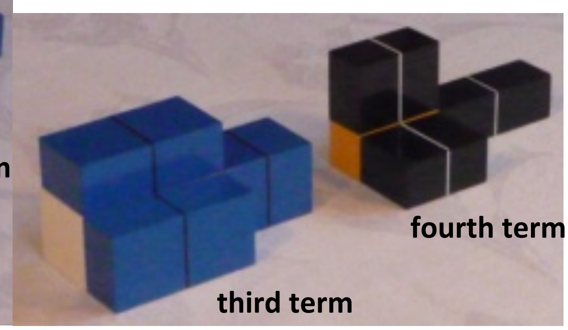
Rearranging the various component prisms according to color, we clearly illustrate the terms of the algebraic expansion of $(a+b)^4$.



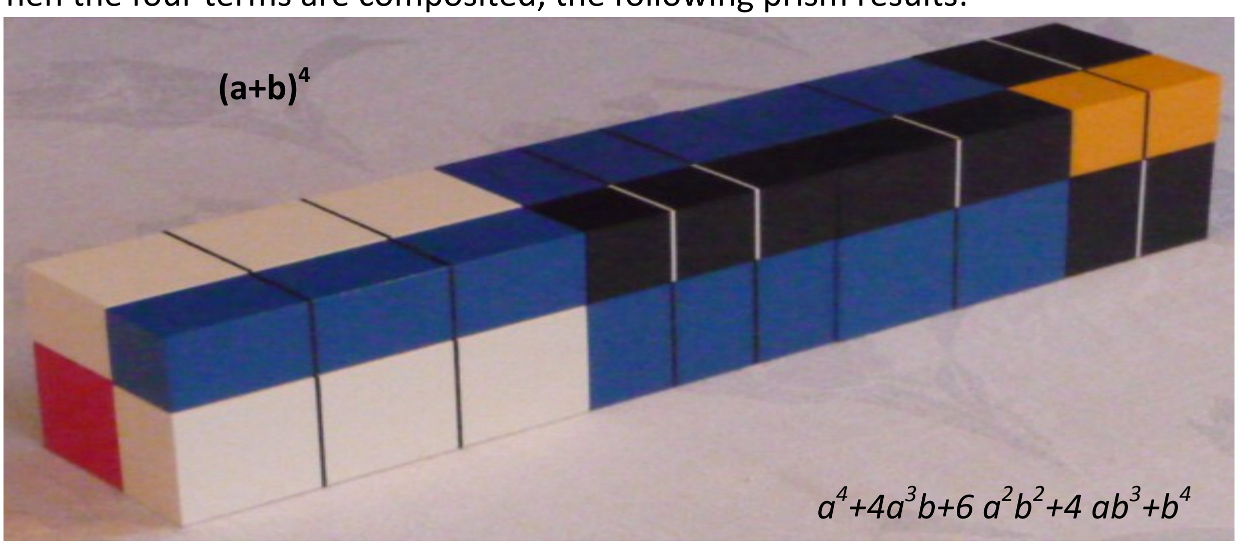
Red	=	(3 cm x 3 cm x 3 cm) x 3 repetitions	=	81 cubic cm	x 1 =	81 = a^4
White	=	(3 cm x 3 cm x 2 cm) x 3 repetitions	=	54 cubic cm	x 4 =	216 = $4\,a^3b$
Blue	=	(2 cm x 2 cm x 3 cm) x 3 repetitions	=	36 cubic cm	x 6 =	216 = $6\,a^2b^2$
Black	=	(2 cm x 2 cm x 3 cm) x 2 repetitions	=	24 cubic cm	x 4=	96 = $4ab^3$
Yellow	=	(2 cm x 2 cm x 2 cm) x 2 repetitions	=	16 cubic cm	x 1 =	16 = b^4

The four terms of the binomial expansion are assembled:



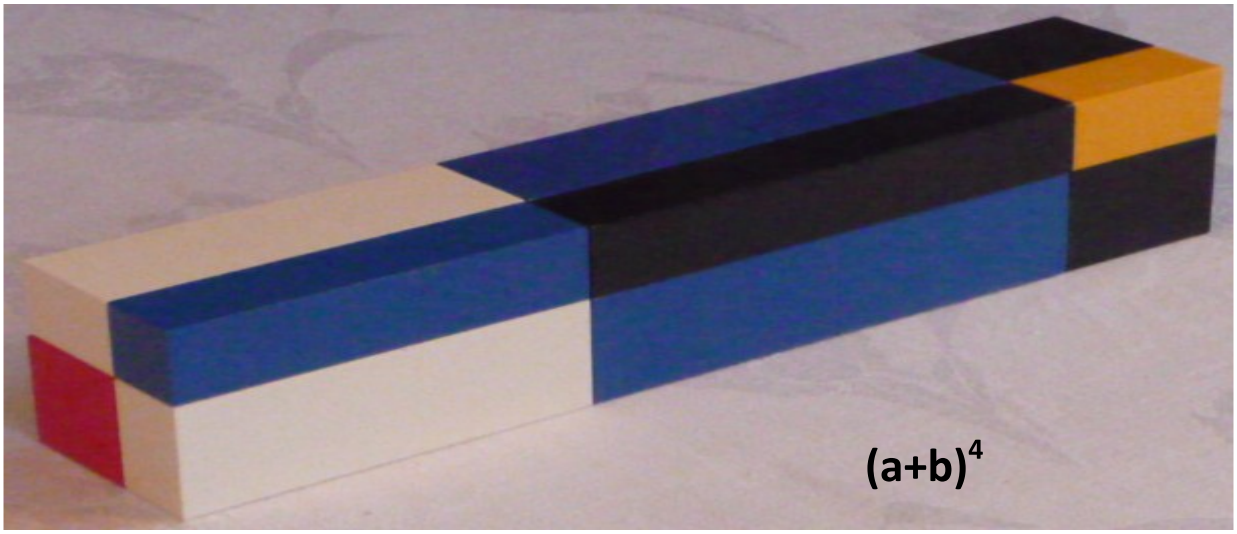


When the four terms are composited, the following prism results:



$(a+b)^4$, Box Four

Comparing boxes Three and Four, we note that in Box Four, as in Box Two, the lines delineating the number of repetitions contained in each object (3 or 2) are removed, leaving only solid-colored prisms for each object. The colors indicating the term of the expansion are preserved.



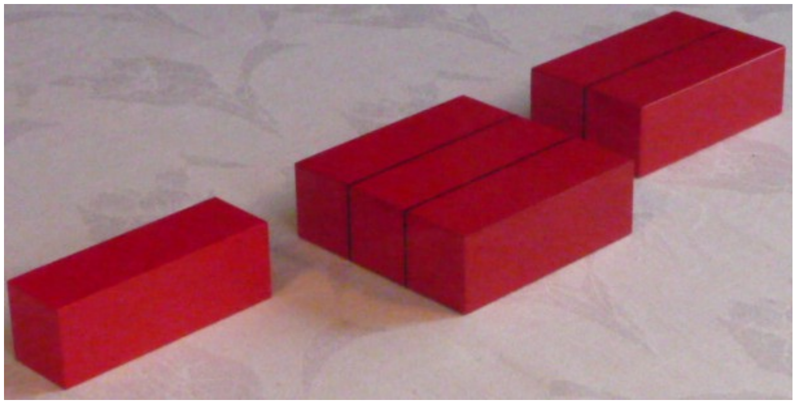
As a final check of the work that we have done, let us examine the specific case built to illustrate the general algebraic expansion. Here, a=3 cm or the abstract numerical value of 3 and b=2 cm or the abstract numerical value of 2. In the final numerical analysis, we have built a prism that has a square cross-section (5cm x 5cm) which is repeated (3+2) times.

The numerical value of the volume of the prism, then, is (5cm x 5cm) x 5 repetitions; the total volume is 625 cubic cm. This can be confirmed by measuring the dimensions of the prism. "These pieces (objects) can be made into a prism that represents the form taken by the fourth power of our duo. Given the measures already decided in our case, the prism … has a square section 5cm x 5 cm, and 25 cm in length."

The Concrete Representation of $(a+b)^5$, Box One

"The fifth power of the combination is built with similar criteria."[1](Montessori, 377)

By analogy, we can represent the fifth power of the binomial by taking the concrete representation of the fourth power of the binomial and repeating it (a+b) times. Once again, we assume the specific case where a=3 and b=2. For example, the first physical object, a^4, is repeated three times in the first term, and twice in the third term:

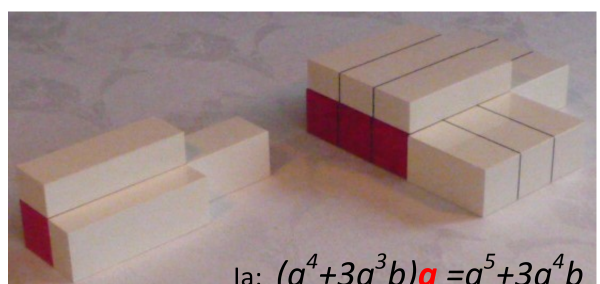


To begin, let us algebraically multiply each term within the fourth power of the binomial, first by a (3) and then by b (2). "The procedure for the construction is analogous to that followed for the fourth power of the binomial, accomplished with these steps:" (Montessori, 377)

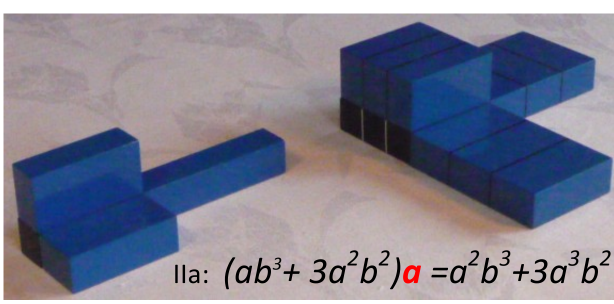
$$(a+b)^5 = (a+b)^4 (a+b)$$
$$=[(a^4+3a^3b)+(ab^3+ 3a^2b^2) + (a^3b + 3a^2b^2)+(b^4 +3ab^3)] \quad [a+b]$$
$$+ [(a^4+3a^3b)+(ab^3+ 3a^2b^2) + (a^3b + 3a^2b^2)+(b^4 +3ab^3)]\ a$$
$$+[(a^4+3a^3b)+(ab^3+ 3a^2b^2) + (a^3b + 3a^2b^2)+(b^4 +3ab^3)]\ b$$

The following illustrations show each term in the expansion. In the photos, the object on the left represents the term from the fourth power of the binomial, while the object on the right represents three or two repetitions of the fourth power of the binomial: the term from the fifth power of the binomial.

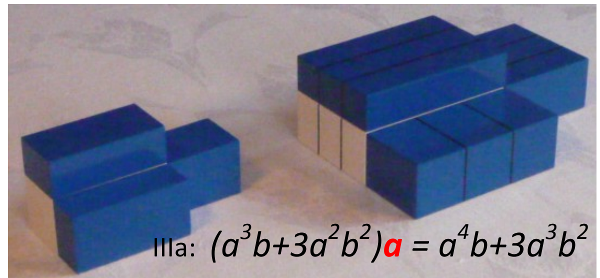
First, we multiply the fourth power of the binomial by ***a*** (**3**):



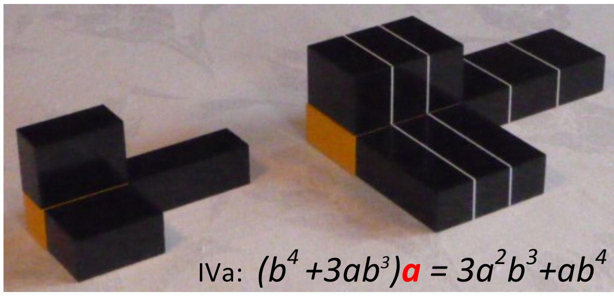
Ia: $(a^4+3a^3b)a =a^5+3a^4b$



IIa: $(ab^3+ 3a^2b^2)a =a^2b^3+3a^3b^2$

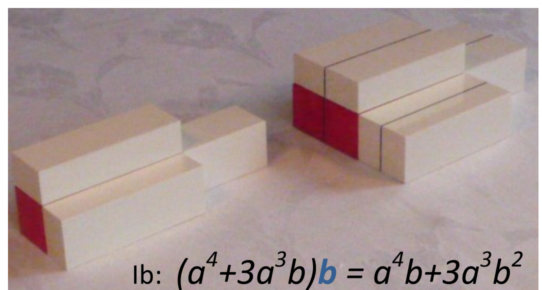


IIIa: $(a^3b+3a^2b^2)\mathbf{a} = a^4b+3a^3b^2$

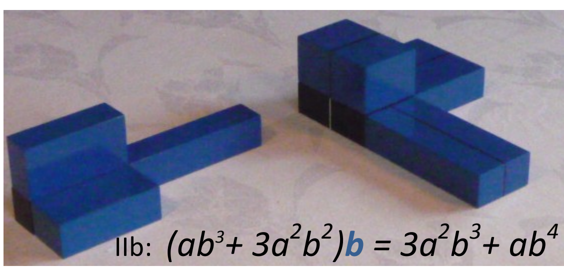


IVa: $(b^4+3ab^3)\mathbf{a} = 3a^2b^3+ab^4$

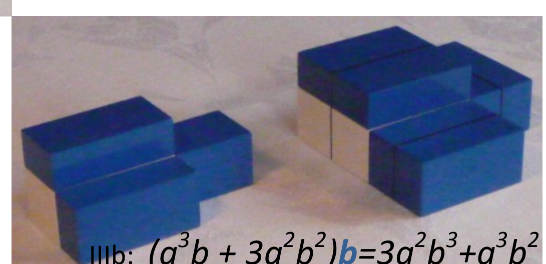
Next, we multiply the fourth power of the binomial by ***b*** (**2**):



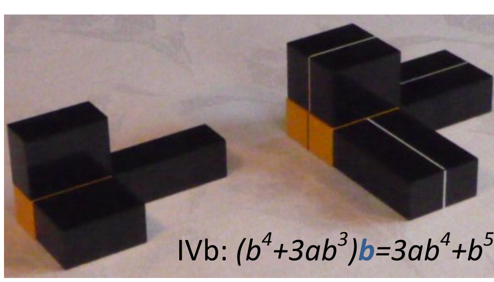
Ib: $(a^4+3a^3b)\mathbf{b} = a^4b+3a^3b^2$



IIb: $(ab^3+ 3a^2b^2)\mathbf{b} = 3a^2b^3+ ab^4$



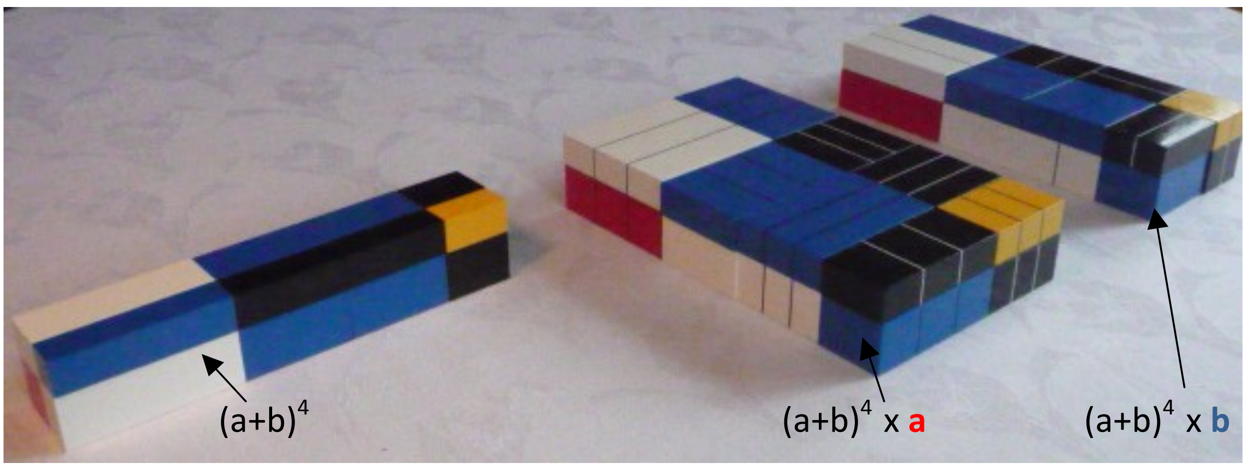
IIIb: $(a^3b + 3a^2b^2)\mathbf{b}=3a^2b^3+a^3b^2$

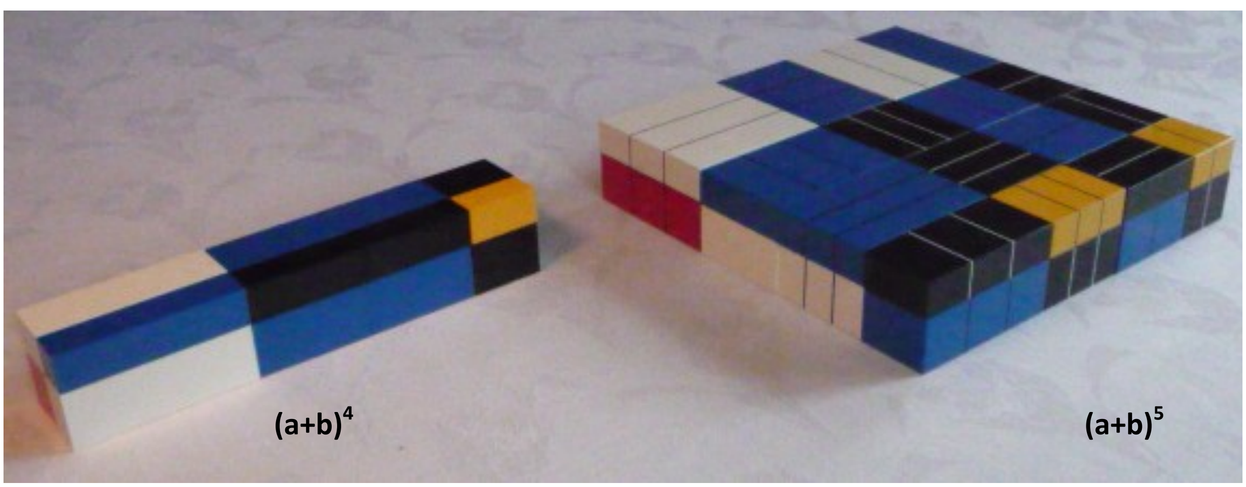


IVb: $(b^4+3ab^3)\mathbf{b}=3ab^4+b^5$

Finally, we assemble each of the parts of the expansions (fourth power x ***a*** and fourth power x ***b***):

Ia:	$(a^4+3a^3b)\mathbf{a}$	=	a^5	$+3a^4b$				
IIa:	$(ab^3+3a^2b^2)\mathbf{a}$	=			a^2b^3	$+3a^3b^2$		
IIIa:	$(a^3b+3a^2b^2)\mathbf{a}$	=		a^4b		$+3a^3b^2$		
IVa:	$(b^4+3ab^3)\mathbf{a}$	=			$3a^2b^3$		$+ab^4$	
Ib:	$(a^4+3a^3b)\mathbf{b}$	=		a^4b		$+3a^3b^2$		
IIb:	$(ab^3+3a^2b^2)\mathbf{b}$	=			$3a^2b^3$		$+ab^4$	
IIIb:	$(a^3b+3a^2b^2)\mathbf{b}$	=			$3a^2b^3$	$+a^3b^2$		
IVb:	$(b^4+3ab^3)\mathbf{b}$	=					$3ab^4$	$+b^5$
	$(a+b)^5$	=	a^5	$+5a^4b$	$+10a^2b^3$	$+10a^3b^2$	$+5ab^4$	$+b^5$

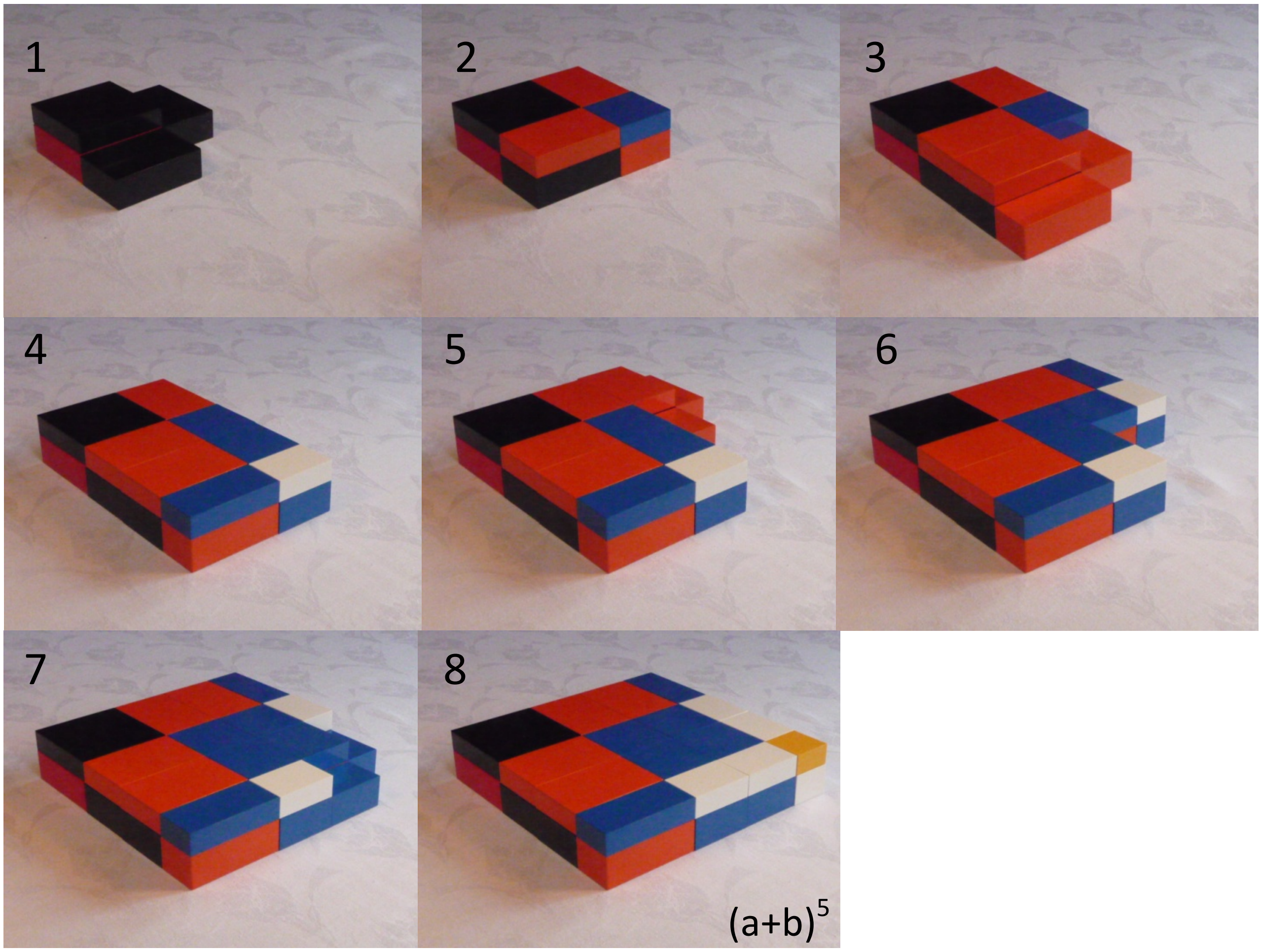




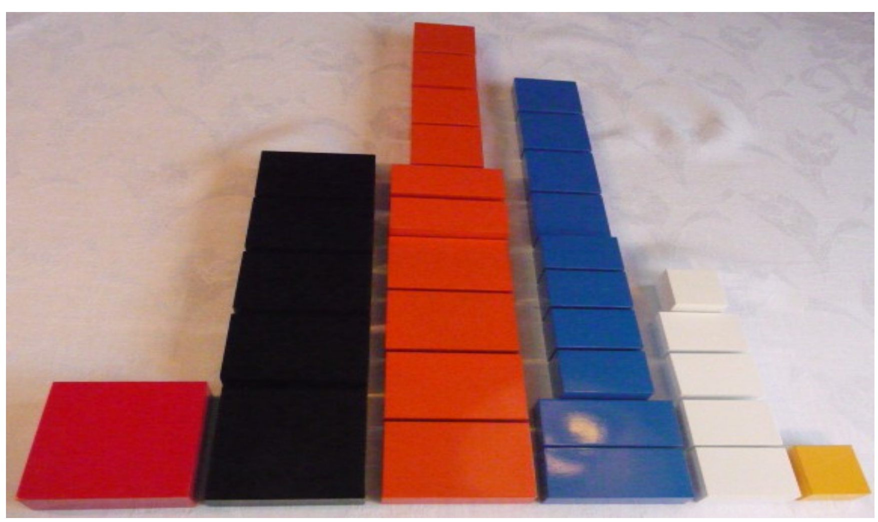
"They result in a large rectangular prism with a square base (with sides measuring) 25 cm each (=9 + 6 + 6 + 4) and with a height of 5 cm (=3+2). This prism is similar to the fifth power, built by exploring in the decimal system… The 'materialization' of the fifth power of the combination is illustrated by … 32 prisms (terms). Of these, none has the form of a cube; they are all rectangular prisms." (Montessori, 377)

The Concrete Representation of $(a+b)^5$, Box Two

A comparison of the various objects from the Fifth Power-Box One to Fifth Power-Box Two, it is easy to see the volumetric correlation. Once again, there is a change in the color schemata. "The prisms of the same algebraic value are then marked with a color (so) that we will have a single prism (for each term), respectively, red - a^5, and yellow - b^5, five black prisms - a^4b, etc." (Montessori, 376) In other words, in Box Two, the colors are again chosen to represent the algebraic terms the prisms. To explore Box Two, we begin by building the fifth power:



Finally, we dissemble the prism and re-sort the individual pieces by color, and the algebraic expansion is beautifully and clearly illustrated. "Its actual construction …is an activity of intelligence and patience that can satisfy both child and adult." (Montessori, 377)



$$(a+b)^5 = a^5 + 5a^4b + 10a^2b^3 + 10a^3b^2 + 5ab^4 + b^5$$